Seven-Sided Star Figures and Tuning Algorithms in Mesopotamian, Greek, and Islamic Texts

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1. *Regular Polygons and Star Figures in Greek and Mesopotamian Texts*

In Euclid's *Elements*, propositions XIII.7-12 deal with the following types of regular n-sided polygons with n = 3, 5, 6, 10: the *equilateral triangle*, the *pentagon*, the *hexagon*, and the *decagon*. (See, most recently, the discussion in Friberg, *Amazing Traces*, Sec. 7.2.) In Hero of Alexandria's *Metrica* I, the sections 17-25 are devoted to rules for the (approximate) computation of the areas of regular n-sided polygons when n = 5, 6, 7, 8, 9, 10, 11, 12. (See Heath, HGM II, 326-329.) Finally, according to Lucian and a scholiast to the *Clouds* of Aristophanes,

> "the triple interwoven triangle, the pentagram, *i.e.* the star-pentagon, was used by the Pythagoreans as a symbol of recognition between the members of the same school, and was called by them Health" (Heath, HGM I, 161).

This means that also the n-sided regular star figure with n = 5, the *pentagram*, was known (and probably studied) by early Greek mathematicians.

One purpose of the present paper is to give a brief survey of all known instances when n-sided regular polygons or star figures occur, in one form or another, on Mesopotamian clay tablets from the 1st, 2nd, and 3rd millennia BC. The results of the survey are presented in tabular form in Fig. 1.1 below. The tabular survey shows n-sided regular polygons with n = 3, 4, 5, 6, 7, and n-sided star figures with n = 5, 7, 8, 12.

In the tabular survey, the following notations are used regarding the type of the texts:

c this is the most likely form of a figure mentioned in a mathematical *table of constants*,

d this is the most likely intended form of a figure shown in a *geometric diagram*,

p this is the most likely form of a figure mentioned in a mathematical *problem text*,

t this is the most likely form of a figure related to entries in a *mathematical table text*.

Note that a substantial part of the examples listed in the tabular survey come from recently published texts in the Schøyen collection (Friberg, MSCT 1), namely: n = 3: examples 1, 2, 3; n = 4: examples 3 and 5; n = 6: example 2. Two come from the Iraq Museum in Baghdad, courtesy F. Al-Rawi, namely both examples with n = 8. Examples 2-3 with n = 7 appear, explicitly and implicitly, in a text discussed quite recently by Horowitz in JANES 30 and by Waerzeggers and Siebes in NABU 2007/2.

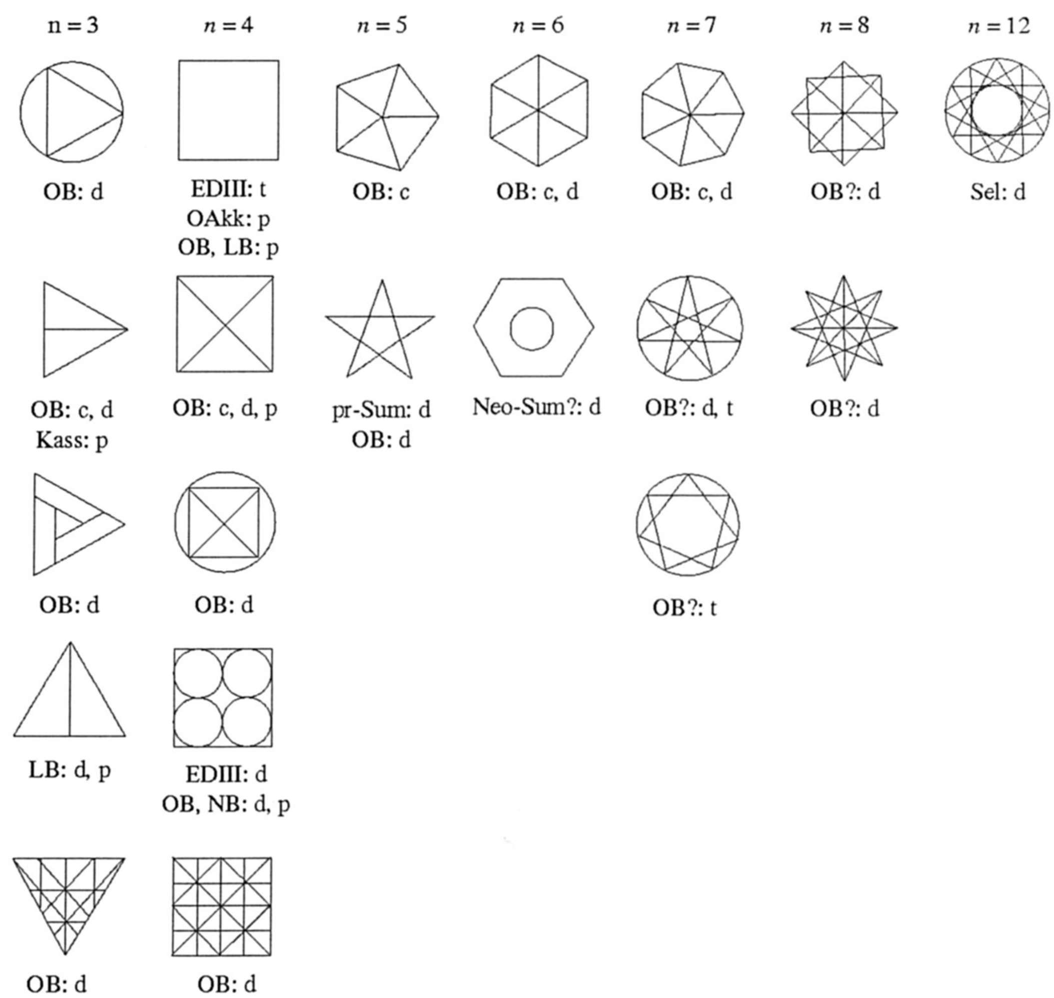

Fig. 1.1. Regular polygons and star figures appearing on known Mesopotamian clay tablets.

[1]) This work has been supported by Stiftelsen Längmanska Kulturfonden.

The more detailed discussion below of the various examples listed in the tabular survey in Fig. 1.1 will start with the examples with the smallest number of sides, proceeding from the several known cases with n = 3 to the single known case with n = 12.

2. n = 3: *Equilateral Triangles*

Equilateral triangles are *three-sided regular polygons*. An equilateral triangle inscribed in a circle appears on the Old Babylonian clay tablet MS 3051 (Friberg, MSCT 1, Fig. 8.1.1; see Fig. 2.1 below). It is likely that the school boy who drew the diagram on the tablet had been given an assignment to compute the areas of the four parts of the divided circle, the equilateral triangle and the three circle segments, when the length of the circumference was given, equal to precisely 1 (· 60 length units).

Indeed, each one of the three circle segments would then be bounded by a circular arc of length 60/3 = 20, as correctly indicated in the diagram. According to a convenient Babylonian convention, the length of the diameter of the circle would be 60/3 = 20, as well, and the length of the radius r = 10. Consequently, the area of the equilateral triangle would be

$A = s/2 \cdot h$, where $h = r + r/2 = 15$ and $s = r \cdot$ sqs. $3 = 10 \cdot$ sqs. 3.[2]

(See Friberg, MSCT 1, Fig. 8.2.4.) However, the incorrect value recorded inside the equilateral triangle in the diagram is 1 52 30, carelessly computed as

$A = h/2 \cdot h = 15/2 \cdot 15 = 7;30 \cdot 15 = 1\ 52;30$.

With departure from this incorrect result, the area of each one of the three circular segments is then computed as

area of segment = (area of circle - area of triangle)/3 = (5 00 - 1 52;30)/3 = 3 07;30/3 = 1 02;30.

This incorrect value is recorded inside each one of the three circular segments.

An equilateral triangle divided into a chain of three trapezoids plus a smaller equilateral triangle is depicted on MS 2192 (Friberg, MSCT 1, Fig. 8.2.2; see Fig. 2.2 below). In this case, the assignment probably was to find the areas of the various parts of the larger triangle when the sides of the two equilateral triangles had the given lengths 1 (· 60) and 10, respectively.

The following curiously formulated entry in the Old Babylonian table of constants G = IM 52916 gives a rule for the computation of the area of an equilateral triangle:

A peg-head (triangle), with an eighth torn out, 26 15 its constant G rev. 7'

What this means is that for an equilateral triangle with the side s the area A is

$A = s/2 \cdot (\text{sqs. } 3)/2 \cdot s =$ (appr.) $s/2 \cdot (1 - 1/8) \cdot s = ;30 \cdot (1 - ;07\ 30) \cdot \text{sq. } s = ;26\ 15 \cdot \text{sq. } s$.

(See Friberg, MSCT 1, Sec. 8.2). This rule is explicitly applied in the Kassite (post-Old-Babylonian) mathematical text MS 3876 (Friberg, MSCT 1, Sec. 11.3), as one of the steps in the correct computation of the weight of the shell of a colossal icosahedron, composed of (6 - 1) · 4 = 20 equilateral triangles of copper, each one of them with the side 3 cubits and the thickness 1 finger (= 1/30 cubit); see Fig. 2.3.

On the obverse of the Old Babylonian tablet TMS 2 (Fig. 5.1 below), the area of an equilateral triangle with the side 30 (one sixth of the area of a regular hexagon with this side) is given as 6 33 45, which is correctly 1/4 of 26 15, the mentioned 'constant' for an

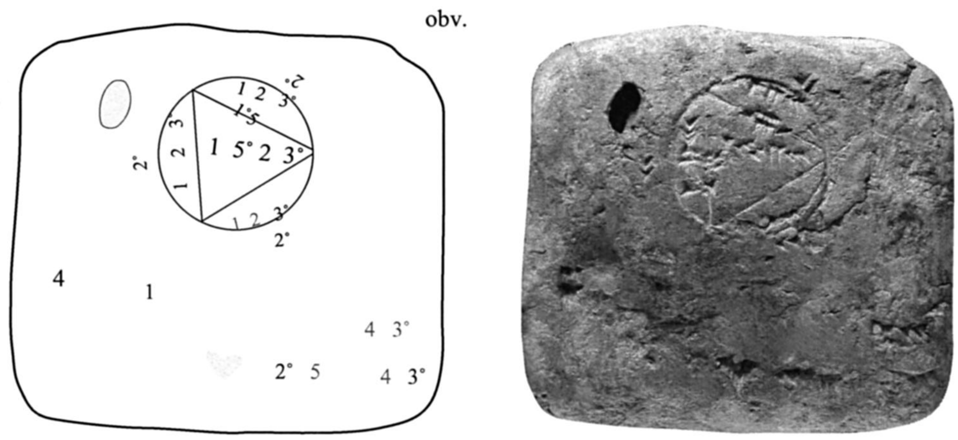

Fig. 2.1. MS 3051. An equilateral triangle inscribed in a circle.

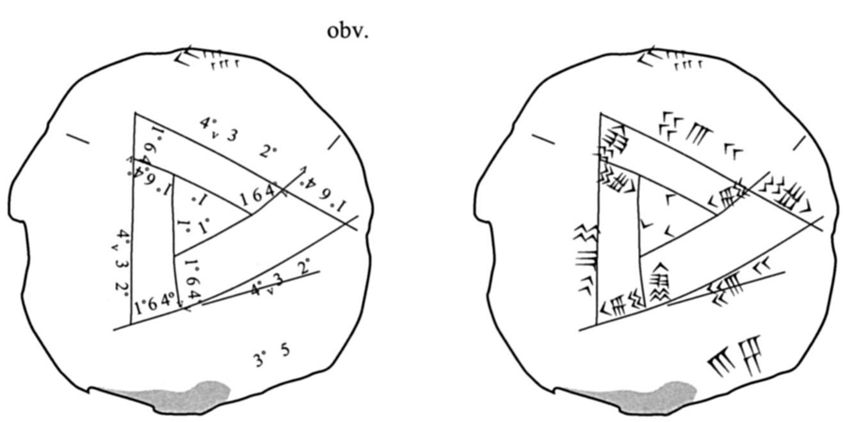

Fig. 2.2. MS 2192. An equilateral triangle divided into three trapezoids and a smaller equilateral triangle.

[2]) The abbreviation sqs. stands for "square-side" or, in modern terms, "square root."

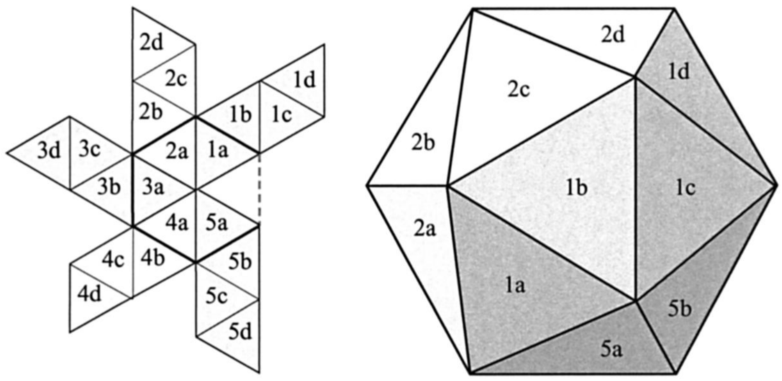

Fig. 2.3. MS 38762. A colossal icosahedron made of (6 - 1) · 4 = 20 equilateral copper triangles.

equilateral triangle, meaning the area of an equilateral triangle with the side 1 (· 60).

In the Late-Babylonian mathematical "recombination text" W 23291 (Friberg, *BaM* 28, 285-286), two rules are given for the computation of the area of an equilateral triangle. In § 4 b, the rule is formulated in the following way:

> 1 peg-head-field, equilateral, that with an 8th torn out, stroke steps of ditto and steps of 26 15 go.

This is clearly a reformulation of the Old Babylonian rule mentioned above. (Here, 'peg-head' means 'triangle,' and 'stroke steps of ditto and steps of 26 15 go' means 'multiply the side length by itself and by 26 15.') The rule is followed by a diagram and a numerical application of the rule; see Fig. 2.4.

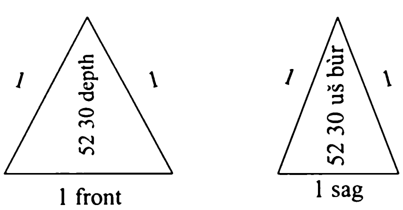

Fig. 2.4. W 23291 § 4 b. An equilateral triangle with the side 1 (· 60) and the height 52;30 = (1 - 1/8) · 60.

Interestingly, the rule in § 4 b, obviously a legacy from Old Babylonian mathematics, is confronted in § 4 c with a more accurate, presumably Late-Babylonian rule:

> 1 peg-head-field, equilateral, that with a 10th and a 30th torn out, stroke steps of ditto and steps of 26 go.

What this means is that for an equilateral triangle with the side s the area A is

$$A = s/2 \cdot (\text{sqs. } 3)/2 \cdot s = (\text{appr.})\ s/2 \cdot (1 - 1/10\ 1/30) \cdot s = ;30 \cdot (1 - ;08) \cdot \text{sq. } s = ;26 \cdot \text{sq. } s.$$

(Thus, in terms of common fractions, the Old and Late-Babylonian approximations to the square side of 3 are, respectively

> sqs. 3 = (appr.) 2 · (1 - 1/8) = 7/4 and sqs. 3 = (appr.) 2 · (1 - 1/10 1/30) = 26/15.

See the thorough discussion of more or less accurate Greek and Babylonian square side approximations in Friberg, *Amazing Traces*, Ch. 16.)

A geometric doodle on the reverse of an Old Babylonian tablet with a single multiplication table on the obverse has the form of a badly drawn "upside-down" equilateral triangle divided into several smaller pieces by two lines parallel to the top and at least two diagonal lines. See Fig. 1.1, bottom (Friberg, MSCT 1, Fig. 8.1.14).

3. n = 4: *Squares*

Squares are *four-sided regular polygons*. The oldest known appearance of squares on any clay tablet from Mesopotamia can be found on a tablet from Šuruppak, dateable to the Early Dynastic IIIa period (c. 2600-2500 BC). The tablet, VAT 12593, is inscribed with a metro-mathematical table of areas of large squares, with side lengths expressed as multiples of the ninda (Friberg, MSCT 1, Fig. 6.1.3).

Also from the Early Dynastic period, but somewhat younger than VAT 12593, and of unknown provenance, is CUNES 50-08-001 (Friberg, MSCT 1, 419-425, Figs. A7.1-2). It is a very large and complex metro-mathematical table of areas of squares, divided into a series of sub-tables with the side lengths of the squares expressed as multiples of the ninda and various fractions of the ninda.

Younger still, from the Early Dynastic IIIb period, is the smaller, but parallel, text A 681 from Adab (Friberg, MSCT 1, 357-60, Fig. A1.4), a table of areas of squares with side lengths expressed as multiples of the cubit.

Some metro-mathematical "field-side-and-area texts" from the Old Akkadian period (c. 2340-2200 BC) contain relatively complicated computations of areas of squares with given side lengths, but no illustrating diagrams (see Friberg, MSCT 1, Sec. A6.2; *id.*, CDLJ 2005: 2, §§ 4.3-4.7).

TSS 77 is a fragment of a round tablet with a diagram of a square with four inscribed circles (see Friberg, *Amazing Traces*, 3.1 6.2.1; Fig. 3.1 below). The

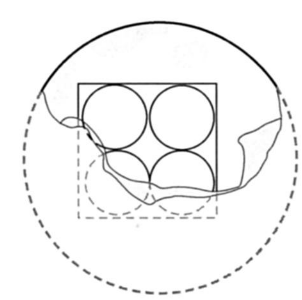

Fig. 3.1. TSS 77. A diagram on a fragment of a round tablet from Old Babylonian Kisurra.

information that this fragment is from Old Babylonian Kisurra, and not from Early dynastic Šuruppak, as was commonly believed earlier, is due to Krebernik, NABU 2006: no. 15.

A square with its diagonals is depicted on the Old Babylonian tablet YBC 7289 (Friberg, MSCT 1, Fig. 16.7.2; Fig. 3.2 below). An accurate approximation is used for the computation of the length of the diagonal when the side of the square has the length 30. The way in which this accurate approximation can have been obtained is discussed in Friberg, *Amazing Traces*, 397.

The copy of YBC 7289 first published by Neugebauer and Sachs in MCT, 42, shows the square standing on one of its corners, with the diagonals horizontal and vertical. Subsequently, the same copy has been republished on numerous occasions in various books and papers, with the square oriented in this way. This is unfortunate, since orienting a square like this is in violation of an easily observed convention in Old Babylonian mathematical texts, according to which representations of triangles, squares, rectangles, trapezoids, *etc.*, always are oriented with one side, the 'front' or 'upper front,' facing left, as for instance, the triangle in Fig. 2.1 above, as well as the hexagon and the heptagon in Fig. 5.1 below. (Note, however, that this Old Babylonian convention does not apply in the case of the hexagon in the *Neo-Sumerian* text MS 1983/2 (Fig. 5.2 below). The correct orientation of the drawing of a square on YBC 7289 has been published before by the present author, for instance in *Amazing Traces*, Fig. 16.7.2. See now also Robson, MAI, 111, Fig. 4.8.

A geometrical doodle on the back of an administrative list from Old Babylonian Mari has the form of a square divided into 16 smaller squares, all with diagonals. See Fig. 1.1, bottom. (Ziegler 1999, no. 37.)

MS 3050 (Friberg, MSCT 1, Fig. 8.2.2; Fig. 3.3 below, left) is a round tablet featuring a square with diagonals, inscribed in a circle. It is hard to make sense of the scattered numbers recorded inside and outside the diagram. It is likely, however, that the text is the result of a school boy's attempt to compute the areas of the various parts into which the circle is divided by the square when, as usual, the length of the circumference of the circle is given. Interestingly, problem # 37 in the demotic mathematical papyrus *P. Cairo* (Friberg, *Unexpected Links*, Sec. 3.1 k) is of precisely this kind, except that the length of the diameter, rather than the circumference, is given.

Two approximations to sqs. 2 (the square-side, alternatively square root of 2), 1;25 and 1;24 51 10, are mentioned in two Old Babylonian tables of constants (TMS 3 and NSe = YBC 7243), in the following way:

1 25	constant of the diagonal of a square	TMS 3 31
1 24 51 10	the diagonal of an equalside	NSe 10

The accurate approximation sqs. 2 = 1;24 51 10 is explicitly mentioned also in YBC 7289 (Fig. 3.2 above), where it is inscribed along the diagonal of the square in the diagram, and where it is used to compute the diagonal d of a square with a side of length 30:

$$d = 1;24\ 51\ 10 \cdot 30 = 42;25\ 35.$$

The value 42 25 35 is recorded just below the diagonal in the diagram.

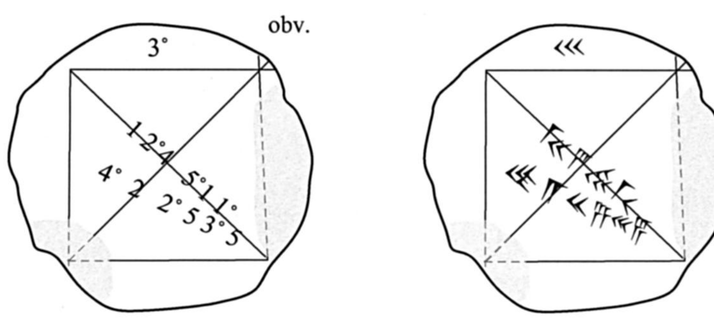

Fig. 3.2. YBC 7289. An Old Babylonian tablet showing a square and its diagonals.

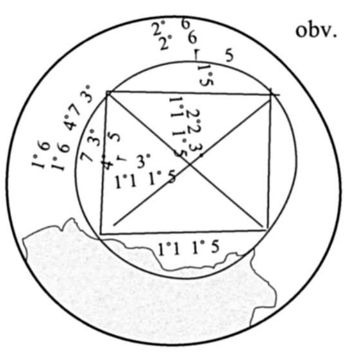

Fig. 3.3. MS 3050. A square with diagonals inscribed in a circle.

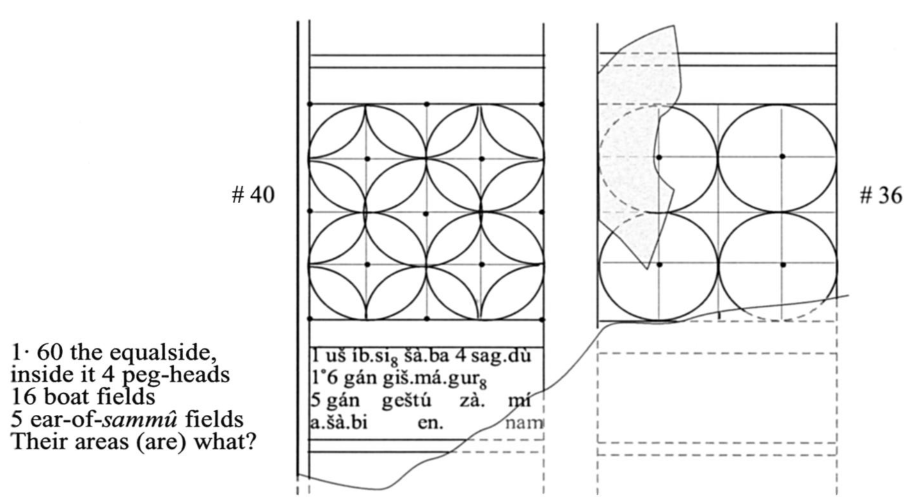

Fig. 3.4. BM 15285 ## 36 and 40. A square divided into various pieces. What is the area of each piece?

It is interesting that 1;24 51 10 is, essentially, the same accurate approximation to sqs. 2 as the one used in Ptolemy's *Syntaxis* I.10! (See Friberg, *Amazing Traces*, Sec. 16.4, and Heath, HGM II, 276-278.) Indeed, the preliminaries to the Table of Chords in Book I.10 of Ptolemy's *Syntaxis* or the *Almagest* (150 AD) include the computation of the side of a regular polygon inscribed in a circle, expressed as a multiple of the 120th part of the diameter of the circle, when the regular polygon in question has 10, 5, 6, 4, or 3 sides. (This is equivalent to computing the chords of 36°, 72°, 60°, 90°, and 120°.) One of the approximations mentioned by Ptolemy is

sqs. 7200 = 84;51 10.

Since, $7200 = 2 \cdot 3600 = 2 \cdot$ sq. 60, the corresponding accurate approximation to sqs. 2 is

sqs. 2 = 84;51 10 / 60 = 1;24 51 10.

The diagram of a square with four inscribed circles in the Early Dynastic text TSS 77 (Fig. 3.1 above) reappears in the well known Old Babylonian geometric theme text BM 15285, in one of 41 exercises where, in each case, a square with the side 1 (· 60) is divided into several pieces by a number of straight or curved lines, and the goal of the exercise is to compute the areas of all the pieces (see Friberg, *Amazing Traces*, Figs. 6.2.2-6.2.3; Robson, MAI, Fig. 2.10).

Although the statement of the problem in BM 15285 # 36 is lost, it is clear that what is asked for is the area of each small piece of the divided square: four circles, one 'ear-of-*sammû*' (a "concave square"), four half concave squares and four quarter concave squares, in the case when the side of the whole square is given as 1 · 60 (ninda).

Fortunately, the corresponding statement in # 40 of a more complicated variant of the same problem is fairly well preserved.

No solutions are offered in the text to the stated problems in BM 15285. In the case of problem # 36 it is impossible to know if the school boy who was asked to find the answer to the problem was supposed to use entries from a geometric table of constants, or if he was supposed to start from scratch. In the latter case, he could find, for instance, the area of the central concave square as *the area of the central small square minus the combined area of four small quarter circles*. In other words, using the well known Old Babylonian rule for the computation of the area of a circle, he could compute the area of the concave square as

A(concave square) = sq. 30 – 4 · 1/4 · ;05 · sq. (3 · 30) = sq. 30 – ;45 · sq. 30 = ;15 · sq. 30 = 3;45.

The story does not end with the Old Babylonian text BM 15285 # 36. Indeed, Robson in *Festschrift Slotsky* has published the Neo-Babylonian tablet BM 47431 (Fig. 3.5 below) with a diagram on the obverse showing four circles inscribed in a square, and with a brief text on the reverse giving an explicit answer to an (unstated) problem of the same type as the explicitly stated problem in BM 15285 # 36.

In spite of the apparent similarity, there are pronounced differences between the *Old Babylonian* text BM 15285 # 36, and the *Neo-Babylonian* text BM 47431. Thus, while the side of the square in the former text is 1 · 60 ninda (1 ninda or 'rod' = c. 6 m), the side of the square in the latter text is 1 · 60 cubits (1 cubit = c. 1/2 m). Moreover, while the sizes of the pieces in the former case were supposed to be expressed in terms of area measure, the sizes of the pieces in the latter case are given in terms of "common seed measure" (see Friberg, *BaM* 28, Sec. 1 and Sec. 6 b). The arbitrarily fixed relation between area measure and common seed measure (csm), expressed either as the amount of seed nominally needed to seed a certain unit of area or, conversely, as the area seeded by a certain capacity unit of seed, can be expressed in various ways, for instance as follows:

1 *pānu* (pi) of seed (csm) corresponds to 3 · sq. (1 · 60 cubits), or

sq. (1 · 60 cubits) corresponds to 2 *sūtu* (bán) of seed (csm).

The following *factor diagram for the Neo-Babylonian system of capacity measure* shows how various units of that system were related to each other:

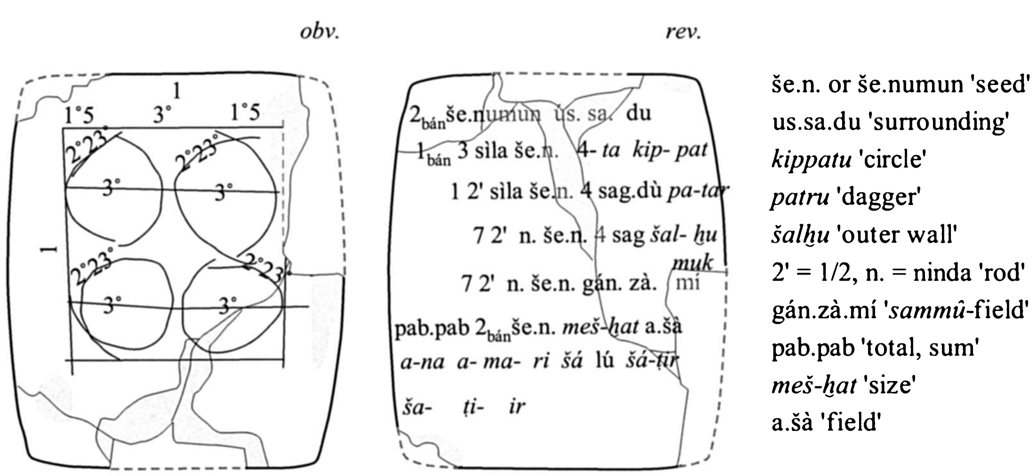

Fig. 3.5. BM 47431. A square divided into various pieces. What are the seed measures of the pieces?

basket c. 1 liter piece of bread

C(NB) : *pānu* $\xleftarrow{6}$ *sūtu* $\xleftarrow{6}$ *qû* $\xleftarrow{10}$ *akalu*

pi bán sìla ninda

Therefore, the answer on the reverse of BM 47431 to the (unstated) question can have been computed in a number of steps, as follows (cf. Robson, *Festschrift Slotsky*, 219-220):

Fig. 4.1. VA 5953. An Old Babylonian mold showing five entangled bearded men forming a pentagram.

1. The common seed measure of the square field surrounding the circles is 2 bán (csm) — rev. line 1
2. The diameter of each one of the inscribed circles is 30 cubits — obv., diagram
   The circumference of each one the 4 circles is (appr.) 3 · 30 cubits = 1 30 cubits
   The combined area of the 4 circles is (appr.) 4 · ;05 · sq. (1 30 cubits) = 45 00 sq. cubits
   The seed measure of the 4 circles is (appr.) ;45 · 2 bán = 1 1/2 bán = 1 bán 3 sìla — rev. line 2
3. The quarter-arc of each one of the circles is 1/4 · 1 30 cubits = 22;30 cubits — obv., diagram
   The area of a concave square with this arc is (appr.) ;26 40 · sq. (22;30 cubits) = 3 45 sq. cubits
   The corresponding seed measure is (appr.) ;03 45 · 2 bán = 7 1/2 ninda
4. The seed measure of the 4 'dagger'-like concave triangles is 4 · 1/2 · 7 1/2 ninda = 1 1/2 sìla — rev. line 3
5. The seed measure of the 4 'outer-wall' concave triangles is 4 · 1/4 · 7 1/2 ninda = 7 1/2 ninda — rev. line 4
6. The seed measure of the central concave square is 7 1/2 ninda — rev. line 5
7. The total seed measure is 1 bán 3 sìla + 1 1/2 sìla + 7 1/2 ninda + 7 1/2 ninda = 2 bán — rev. line 6

Note the use in these computations of the following well known 'constants' (igi.gub):

5	the 'constant for a circle'
26 40	the 'constant for an (ear-of-)*sammû*-field (concave square).

The constant 5 for a circle appears in 7 Old Babylonian tables of constants (see Robson, *Mesopotamian Mathematics*, Sec. 3.1) and also in the Neo-Babylonian table of constants CBS 10996 (see Sec. 11 below). The constant 26 40 for a concave square appears in 4 Old Babylonian tables of constants (see Robson, *Mesopotamian Mathematics*, Sec. 3.7). It is likely that it also appeared in the now lost part of the Neo-Babylonian table of constants CBS 10996 (Sec. 11 below).

4. n = 5: *Regular Pentagons and Pentagrams*

An entry in TMS 3, an Old Babylonian table of constants from the ancient city Susa (in western Iran), mentions in the following way the 'constant' for a '5-front' (a regular *pentagon*), meaning the area of the 5-front when the side length is 1 (· 60):

1 40	igi.gub	šà	sag.5
1 40	the constant of a 5-front		
			TMS 3 26

The value 1 40 is easily explained: If the side length of the pentagon is 60, then the length of the circumscribed circle is (appr.) 5 · 60. Therefore, the radius of the circumscribed circle is (appr.) 5 · 10 = 50. Consequently, the 5-front can be divided into five triangles, all of which have one side of length 60 and two sides of length 50. The area of one such triangle is 30 · 40 = 20 (· 60). Hence the area of the 5-front is 1 40 (· 60).

There is no other known occurrence of a regular pentagon in a Mesopotamian text. However, VA 5953 (Friberg, *Amazing Traces*, Fig. 7.9.7; see Fig. 4.1 above) is an Old Babylonian mold showing in relief five entangled bearded men forming a 5-sided star figure (a *pentagram*) enclosing a regular pentagon.

A much older Mesopotamian example of a picture of a pentagram is UE 3, 398 (Friberg, *Amazing Traces*, Fig. 7.9.2; Fig 4.2 below), a copy of a seal imprint from a layer beneath the royal cemetery at Ur, dated to the proto-Sumerian Jemdet Nasr period around the beginning of the 3rd millennium BC. Note the appearance in the lower left corner of a pentagram drawn in one uninterrupted line.

Fig. 4.2. UE 3, 398. A pentagram appearing in a seal imprint from the proto-Sumerian Jemdet Nasr period.

Actually, a pentagram appears in this seal imprint sooner for calligraphic than for artistic reasons. Indeed, in the proto-cuneiform script used in Mesopotamia in the Jemdet Nasr period, the sign UB had the form of a pentagram. (Several of the other images in the seal imprint in Fig. 4.2 are also proto-cuneiform signs.) Just as in this seal imprint, UB appears frequently in proto-cuneiform texts from Jemdet Nasr together with the sign AB. An example, borrowed from Englund and Grégoire, MSVO 1, is shown in Fig. 4.3.

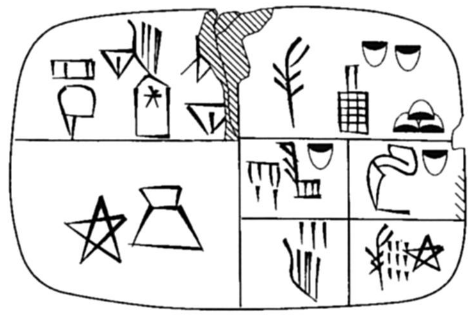

Fig. 4.3. Englund and Gregoire, MSVO 1, 220 = IM 55587. A proto-cuneiform text from the proto-Sumerian Jemdet Nasr period.

5. n = 6: *Regular Hexagons*

Another entry in the Old Babylonian table of constants TMS 3 mentions the 'constant' for a '6-front' (a regular *hexagon*), meaning the area A of the 6-front when the side length is 1 (· 60):

2 37 30	igi.gub	šà sag.6	
2 37 30	the constant of a 6-front		TMS 3 27

In view of the discussion above of the case $n = 3$, this value can be explained as

$A = 6 \cdot 1/2 \cdot (1 - 1/8) \cdot \text{sq. } 60 = 6 \cdot 26\ 15 = 2\ 37\ 30.$

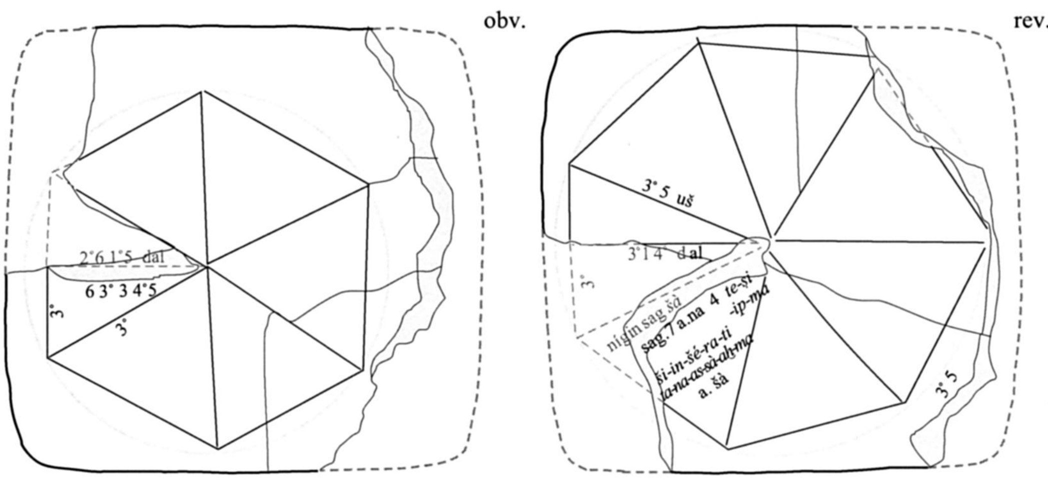

Fig. 5.1. TMS 2. An Old Babylonian tablet from Susa with images of a hexagon and a heptagon.

A regular hexagon with sides of length 30 is depicted on the obverse of TMS 2 (Friberg, MSCT 1, Fig. 8.2.15; Fig. 5.1 below), an Old Babylonian tablet from Susa. As indicated by the number 6 33 45 recorded in the left-most equilateral sub-triangle, the area of a regular hexagram with this side length could be computed as follows:

$A = 6 \cdot 1/2 \cdot (1 - 1/8) \cdot \text{sq. } 30 = 6 \cdot 26\ 15 \cdot 1/4 = 6 \cdot 6\ 33;45\ (= 39\ 22;30).$

MS 1983/2 (Friberg, MSCT 1, Figs. 8.1.12, 8.2.14; Fig. 5.2 below) is a large fragment of a mathematical tablet, probably from the Neo-Sumerian Ur III period.

The tablet is inscribed on the obverse with a diagram showing a trapezoidal field divided into five parallel stripes with areas forming an arithmetic progression, and on the obverse (according to a likely reconstruction) with the image of a regular *hexagon* with a circle in the middle, probably some kind of geometric assignment.

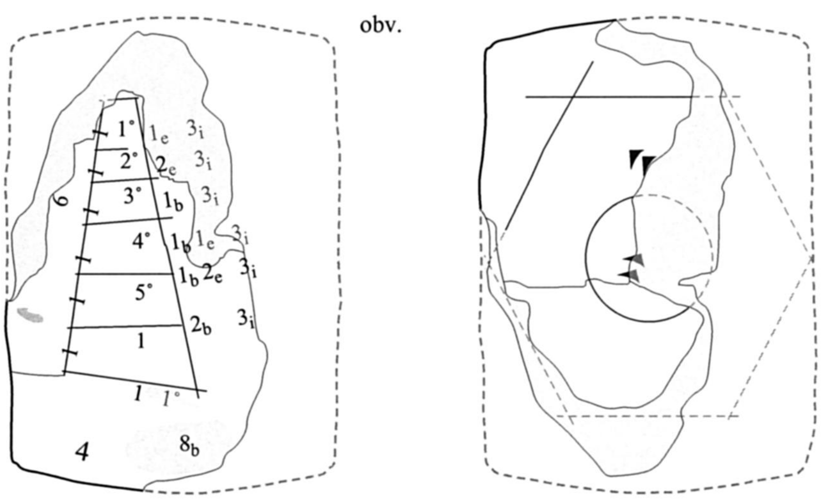

Fig. 5.2. MS 1983/2. A mathematical tablet, probably from the Neo-Sumerian Ur III period.

6. n = 7: *Regular Heptagons and Two Kinds of 7-Sided Star Figures*

A third entry in the Old Babylonian table of constants TMS 3 mentions the 'constant' for a '7-front' (a regular *heptagon*), meaning the area A of the heptagon when the side length is 1 (· 60):

3 41	igi.gub šà sag.7	
3 41	the constant of a 7-front	TMS 3 28

There is also an image of a regular heptagon on the reverse of TMS 2, the tablet from Susa shown above in Fig. 5.1. The heptagon has the given side length 30, and therefore the radius r of the circumscribed circle can be computed as

r = (appr.) 1/6 · 7 · 30 = 35.

That is, of course, why the value 35 uš '35, the length' is recorded above a radius of the heptagon on the reverse of TMS 2.

One would now expect to find the total area of the upper triangle or of the whole heptagon recorded in the diagram. That is not the case. Instead one finds a somewhat cryptic inscription, interpreted as follows by Robson in *Mesopotamian Mathematics*, 49:

[nigin sag *šà*] sag.7
a.na 4 *te-ṣi-ip-ma*
ši-in-šé-ra-ti ta-na-as-sà-aḫ-ma a.šà
[*The square of the front*] (the side) of the 7-front
by 4 you repeat, then
the twelfth you tear out, then the field (the area).

What this means is that the area of a heptagon with the side s can be computed as

A = (appr.) 4 · sq. s – 1/12 of 4 · sq. s = 4 · sq. s – ;20 · sq. s = 3;40 · sq. s.

Thus, you get the area of the heptagon if you multiply the square of the front by 4, and reduce the result by a twelfth of its value. The computation rule is a handy variant of the more formal rule A = sq. s · 3;40. Compare the entry '3 41 the constant of a 7-front' in TMS 3 28, where the value 3 41 probably had been computed as follows. When s = 1 (· 60), then

A = (appr.) 7 · 30 · sqs. (sq. 1 10 – sq. 30) = 7 · 30 · 20 · sqs. 10 = (appr.) 7 · 10 · 3 10 = (appr.) 3 41 (· 60).

CBS 1766. *Description of the Diagram and the Table*

A photo of CBS 1766 was first published by Hilprecht in his *Explorations* on p. 530, where the text was loosely characterized as an "astronomical tablet from the Temple Library."

Subsequently, CBS 1766 was largely ignored for more than a century, until Horowitz saved it from oblivion by republishing a photo of it in JANES 30 pp. 37-53, together with a transliteration of the text and an attempted interpretation of it.

A year later, in NABU 2007/2, C. Waerzeggers and R. Siebes suggested alternative transliterations of several crucial words in the text and an interesting new interpretation of the whole text. The discussion below of the text is largely based on their new interpretation.

The reverse of the tablet is completely destroyed. On the obverse, in a box in the upper left corner, which is fairly well preserved, there is a diagram of a 7-sided star figure (a *heptagram*), drawn in one uninterrupted line consisting of a chain of 7 straight lines of equal length), and in the lower half of the obverse there is a numerical table. The rest is empty. By luck, although there are some missing parts of the clay tablet, nothing seems to have been lost of the inscription on the lower half of obverse. See Fig. 6.1 below.

The star figure on the obverse of CBS 1766 is inscribed in a double circle. Both the star figure and the circles are drawn without much care, and without the use of compass and ruler. The seven points of the star figure are numbered, from 1 to 7, and briefly labelled in the following way:

1	*qú-ud-mu*	the foremost
2	⌜*ša*⌝-*mu-šum*	the next
3	[*šal-šu qat-nu*]	[the third, thin]
4	*e-ba-nu*	the one constructed by (the god) Ea
5	*ḫa-an-šu*	the fifth
6	*re-*⌜*bi*⌝ *uḫ-ri*	the fourth behind
7	*šal-šu* ⌜*uḫ-ri*⌝	the third behind.

These are known names for seven strings of the Mesopotamian harp or lyre. See the further discussion of this topic in the 10th section of the present paper.

Note: In Babylonian mathematical texts, the essential components of geometric figures are their straight or curved segments, never their vertices. Cf. the discussion of Babylonian "metric algebra diagrams" *vs.* Greek "lettered diagrams" in Friberg, *Amazing Traces*, Sec. 1.1. Therefore, it is likely that the numbers inscribed around the star figure in CBS 1766 should be understood as components of *number pairs* defining the seven sides of the star figure, not as single numbers defining the seven points of the star figure!

The table on the obverse of CBS 1766 contains 11 columns, organized as follows: Column 1 is empty, columns 2-3 are both inscribed with 7 lines of number pairs. Column 4 is again empty, while columns 5-6 are inscribed with only 1 line each of number pairs. Apparently, the writing of numbers in the table was interrupted here and never finished. Indeed, the remaining columns are empty, except for the last column, which contains traces of a few words (not numbers). The interruption may have been unintentional, but another possibility is that the text in its present state was an assignment, and that a school boy had been asked to fill in the remaining numerical parts of the table, which he never did.

There is a (somewhat) readable line of text as a heading over columns 1-4 in the table. The headings

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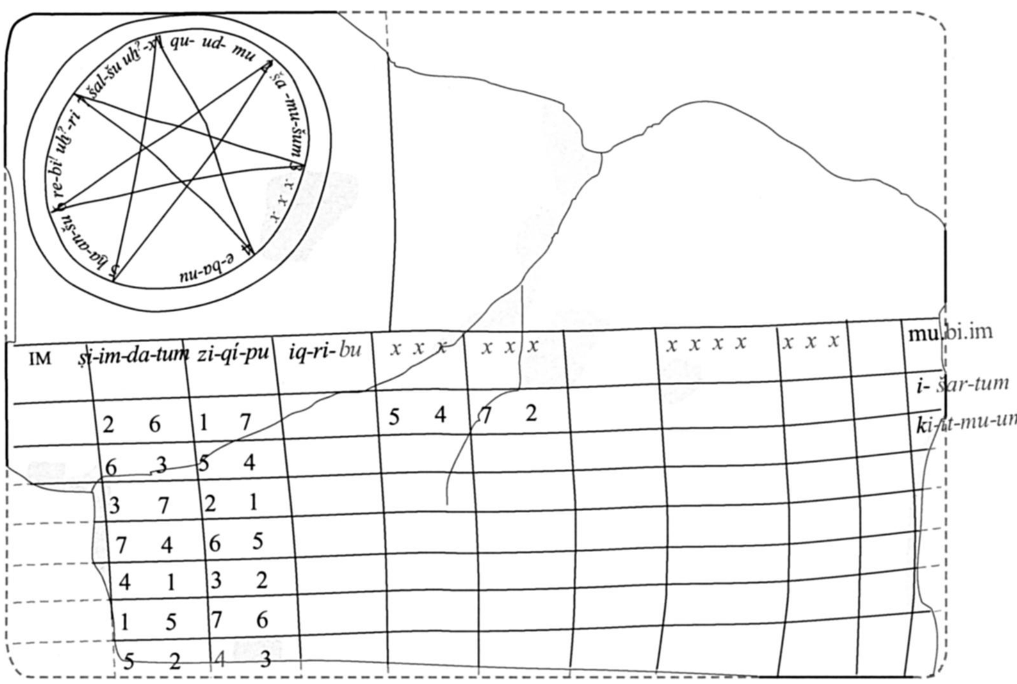


Fig. 6.1. CBS 1766. A 7-sided star figure and a numerical table. Photo and conform transliteration. Published here with the kind permission of G. Frame.

over the other columns of the table, if any, are unreadable, except for the heading over the non-numerical entries in the last column, but that too is badly readable.

The line of text above columns 1-4 is read as follows by Horowitz, JANES 30, pp. 37-53:

> IM *ṣi-im-da-tum zi-qí-pu iq-r*[*i-bu*? ...] or *ik-ta*[*l-du*? ...]
> pairs of altitudes which appro[ached ...] or rea[ched ...].

No serious attempt was made by Horowitz to explain the meaning of this line of text.

CBS 1766. *A Geometric Explanation of the Number Pairs in the Table*

In column 2, the first inscribed column on CBS 1766, the second number in each pair is equal to the first number in the next pair. Thus, the first pair 2, 6 is followed by the second pair 6, 3, the third pair 3, 7, and so on. To make sense of this observation, assume that

> In column 2, the pair 2, 6 stands for the side in the 7-sided star figure which goes from the point labelled 2 to the point labelled 6, the next pair 6, 3 stands for the side of the star figure which goes from the point labelled 6 to the point labelled 3, and so on.

See Fig. 6.2, top. Thus, the seven number pairs in column 2 can be interpreted as

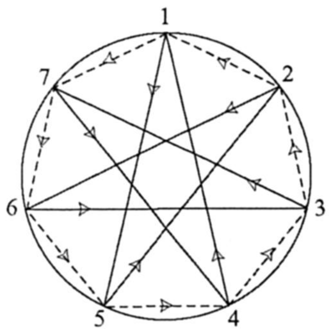
> A description of how to draw the whole 7-sided star figure by use of an uninterrupted chain of straight lines running through the points labelled 2, 6, 3, 7, 4, 1, 5, 2, in this order.

Now, if this is the correct interpretation of the seven number pairs in the first inscribed column on CBS 1766, what is then the corresponding interpretation of the seven number pairs in the second inscribed column, column 3? It ought to be

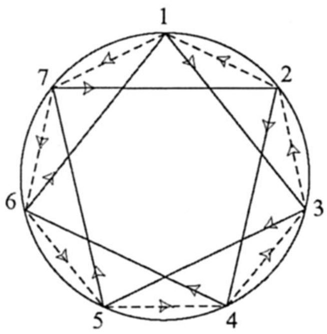
> A description of how to draw a certain diagrammatic figure by use of a set of (non-connected) straight lines running through the pairs of points labelled (1, 7), (5, 4), (2, 1), (6, 5), (3, 2), (7, 6), and (4, 3).

See again Fig. 6.2, top. This diagrammatic figure is clearly a regular *heptagon*, that is a polygon with 7 equal sides, which can be inscribed in a circle.

Assuming that this interpretation is correct, it remains to explain what the precise relation is between



consecutive sides of the 7/3 star figure	parallel, nearby sides of the heptagon
2, 6	1, 7
6, 3	5, 4
3, 7	2, 1
7, 4	6, 5
4, 1	3, 2
1, 5	7, 6
5, 2	4, 3



parallel, distant sides of the heptagon	consecutive sides of the 7/2 star figure
5, 4	7, 2
[7, 6]	[2, 4]
[2, 1]	[4, 6]
[4, 3]	[6, 1]
[6, 5]	[1, 3]
[1, 7]	[3, 5]
[3, 2]	[5, 7]

Fig. 6.2. Suggested geometric explanation of the four preserved tables of number pairs on CBS 1766.

the two number pairs in each line of columns 2-3. In particular, what is in the case of the first line of the two columns the relation between the side from 2 to 6 in the star figure and the side from 1 to 7 in the heptagon? The answer is obvious, since the two sides are parallel to each other and go in the same direction. Similarly, in the case of the second line, the side from 6 to 3 in the star figure is parallel to the side from 5 to 4 in the heptagon and goes in the same direction. And so on. (It is assumed here that, like the star figure, *the heptagon is drawn in one uninterrupted chain of straight lines*, *counter-clockwise*, so that each side of the heptagon has a given direction.) Therefore, the number pairs in the first two inscribed columns on CBS 1766 quite explicitly demonstrate that

> For each side in the 7-sided star figure there is a parallel side in the heptagon going in the same direction.

Note that because of its manner of construction, the 7-sided star figure, like the regular heptagon, can be inscribed in a circle, that is, both figures are "cyclic." The diagram on the obverse of CBS 1766 shows the 7-sided star figure being inscribed in a double circle, but that is probably just an embellishment (if not to accommodate further text).

In Fig. 6.2, top, the 7-sided star figure is called a "7/3 star figure." What this means is that in order to draw the star figure, you can start at one of the numbered points, say the point 2, proceed from there counter-clockwise to the 3rd point along the circle, which is then the point 6, draw a straight line from 2 to 6, and then repeat the procedure until the diagram returns to the starting point. The result will be the 7-sided star figure running through the points 2, 6, 3, 7, 4, 1, 5, 2.

In a similar sense, the heptagon itself can be understood as a "7/1 star figure."

This observation immediately leads to the following question: What is then a "7/2 star figure?" The answer is demonstrated by the diagram in Fig. 6.2, bottom. Start, say, at the point 7 and proceed from there counter-clockwise to the 2nd point along the circle, which is then the point 2, and draw the straight line from 7 to 2. Repeat the process until the diagram returns to the starting point. The result will be a new kind of 7-sided star figure running through the points 7, 2, 4, 6, 1, 3, 5, 7, in this order. Now observe that the side from 5 to 4 in the heptagon is parallel to the side from 7 to 2 in the 7/2 star figure. It is likely that this observation explains the two pairs 5, 4 and 7, 2 in the first line of columns 5-6 on CBS 1766. Consequently, it is likely that the unfinished second pair of inscribed columns was intended to show, quite explicitly, that

> For each side in the 7/2 star figure there is a parallel side in the heptagon going in the same direction.

See again Fig. 6.2, bottom. (It is not clear why the pair 5, 4 precedes the pair 7, 2, so that the order of the two columns is not the same in the case of the 7/2 star figure as in the case of the 7/3 star figure.)

Since the numerical table on CBS 1766 was left unfinished, it is impossible to know what a third pair of inscribed columns (columns 8-9) could have contained. Maybe they would have been concerned with a "7/4 star figure." Now, it is easy to see that a 7/4 star figure is identical with a 7/3 star figure running in the opposite direction, through the points 2, 5, 1, 4, 7, 3, 6, 2. Similarly, a 7/5 star figure is just a 7/2 star figure running in the opposite direction, through the points 7, 5, 3, 1, 6, 4, 2, 7, and a 7/6 star figure is the same as a heptagon running in the opposite direction, through the points 1, 2, 3, 4, 5, 6, 7, 1.

It is now time to return to the meaning of the line of text over columns 1-4. It is suggested here, very tentatively, but in essential agreement with one of the possibilities suggested by Horowitz, that there are three distinct headings in columns 1, 2, and 3 (spilling over into column 4), and that these headings should be read as, respectively,

col. 1: ⌜IM⌝	directions?
col. 2: *ṣi-im-da-tum*	pairs
col. 3: *zi-qí-pu iq-r*[*i-bu*]	the stakes are close together.

The "pairs" mentioned in the heading over col. 2 can, of course, be understood as the number pairs from 2, 6 to 4, 2 specifying the 7 sides of the 7/3 star figure.

Finally, since stakes are usually straight (and upright), it is possible that in this text the term 'stakes' stands for 'straight lines,' and that the meaning of the

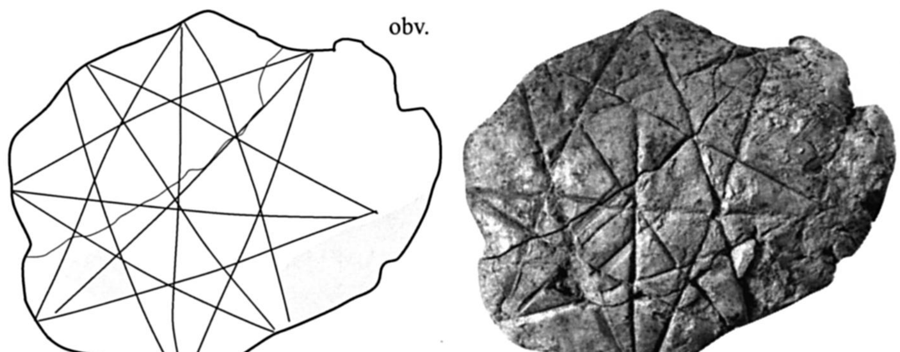
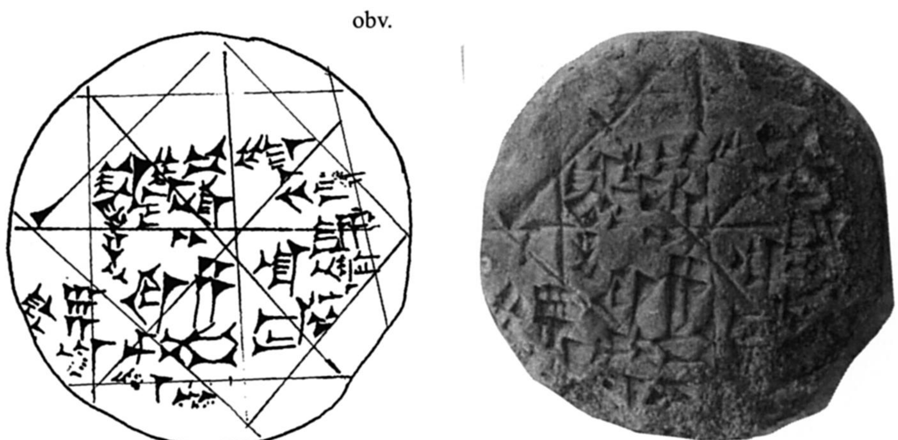


Photo: F. Al-Rawi

Fig. 7.1. IM 51979. An Old Babylonian(?) tablet showing an 8/3 star figure with its diagonals.



Copy and photo: F. Al-Rawi

Fig. 7.2. An Old Babylonian tablet from Haddad with an 8/2 star figure and scribbled cuneiform signs.

line of text over column 3 is that the straight lines making up the sides of the 7/3 star figure are *nearby* the straight lines parallel to them, which make up the sides of the heptagon, and which are specified by the number pairs in column 3. See Fig. 6.2, top.

In contrast to the situation in Fig. 6.2, top, the situation in Fig. 6.2, bottom, is that the straight lines making up the sides of the heptagon are distant from the straight lines parallel to them, which make up the sides of the 7/2 star figure, and which are specified by the number pairs in column 6.

7. n = 8: *Two Old Babylonian? Tablets with Two Kinds of 8-Sided Star Figures*

IM 51979 (Friberg, *Amazing Traces*, Fig. 7.8.2; Fig. 7.1 above) is a roughly made tablet, possibly Old Babylonian, inscribed exclusively with a diagram showing an *8/3 star figure*, drawn in one uninterrupted line, and its 4 diagonals.

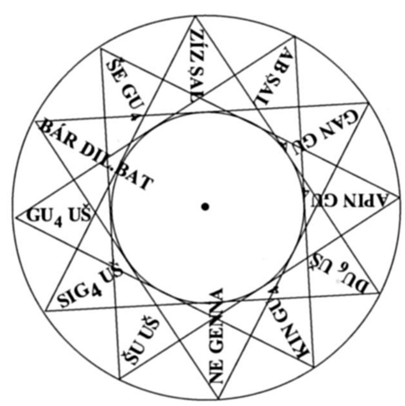
A previously unpublished Old Babylonian tablet from Haddad (Fig. 7.2) is inscribed with an *8/2 star figure* and its 4 diameters. The 8/2 star figure cannot be drawn with one uninterrupted line. Instead it is composed of 2 squares.

The meaning of the many scribbled cuneiform signs inside the figure on the obverse of this tablet, and similarly scribbled signs on the reverse, is not at all clear.

8. n = 12: *A 12-sided Star Figure in a Seleucid Astrological Text*

The Seleucid astrological text O 176, in which *a 12-sided (and 12-pointed) star figure* appears in an isolated position, was first published by Thureau-Dangin in TCL 6, text 13. A commentary appeared much later, in Rochberg-Halton, ZA 77.

Names of months and planets are inscribed in the 12 points of the star figure (see Fig. 8.1 below). However, there is no obvious connection



(Abbreviated) month names:

BÁR	(I)	ŠU	(IV)	DU$_6$	(VII)	AB	(X)
GU$_4$	(II)	NE	(V)	APIN	(VIII)	ZÍZ	(XI)
SIG$_4$	(III)	KIN	(VI)	GAN	(IX)	ŠE	(XII)

(Abbreviated) planet/god names:

DIL.BAT (Venus/Ishtar) UŠ (Saturn/Ninurta) GENNA(?) (also Saturn?)
GU$_4$ (Mercury/Nabû) ṢAL (Mars/Nergal)

Fig. 8.1. O 176. A twelve-sided star figure with inscribed and circumscribed circles, month names and planet (or god) names. The astrological meaning of this diagram is unknown.

between these names of months and planets on one hand and the astrological text on the tablet on the other hand.

The 12-sided star figure can be more distinctly characterized as a *12/4 star figure*, since each side in the star figure goes from one of the 12 points of the star to a point 4 steps removed from it along the circle. The star figure cannot be drawn with one uninterrupted line. Instead, it is composed of 4 equilateral triangles. Note the presence of both a circumscribed and an inscribed circle.

9. *A Starlike Doodle on the Reverse of an Early Dynastic IIIa Lexical Text*

Loosely associated with the theme of polygons and star figures on Mesopotamian clay tablets is a doodle on the reverse of the lexical text VAT 9128 from Early Dynastic IIIa Shuruppak, 2600-2500 BC. The photo of the reverse of the clay tablet in Fig. 9.1 below (available online at cdli.ucla.edu, P010673) shows a last half-line of the lexical text, and as space fillers on the otherwise empty reverse a drawing of a grazing antelope and a doodle in the form of a kind of non-regular 6-sided star figure with embellished "diagonals." The way in which the doodle was drawn is illustrated in Fig. 9.2 below.

Two other examples of quasi-mathematical space-filling doodles on the reverses of similar texts from Early Dynastic IIIa Shuruppak are shown in Friberg, MSCT 1, Figs. A6.21-22.

Fig. 9.1. VAT 9128. The reverse of a lexical text from ED IIIa Šuruppak with a drawing and a doodle.
The photo is reproduced here with the kind permission of Bildagentur für Kunst, Kultur und Geschichte.

10. *Names of Strings, Intervals, and Modes on an Instrument with 9 Strings*

For the readers' convenience, in this section of the paper are brought together some well known facts about cuneiform texts mentioning names of strings, intervals, and scales on a harp or lyre with 9 strings. A knowledge of such items is necessary for the proper understanding of the meaning of the 7-sided star figure and the numerical table on CBS 1766.

The names of the 9 strings, in both Sumerian and Akkadian, are mentioned in columns i-ii of UET VII 126, a Neo-Babylonian fragment of a copy of the 32nd tablet of the lexical series *Nabnītu* 'creation' (Fig. 10.1). See Kilmer, *Festschrift Landsberger*.

According to this text, five of the nine strings are counted from the front (of the string instrument), while the remaining strings are counted from the rear.

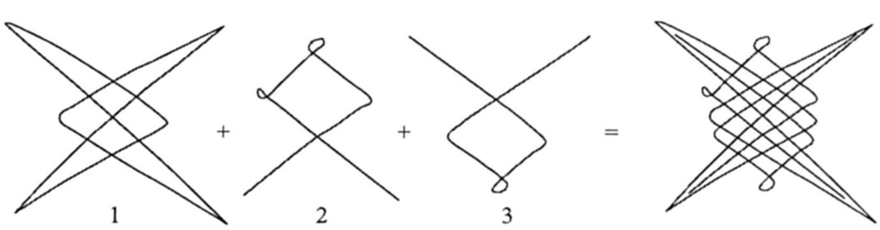


Fig. 9.2. VAT 9128. The construction in three steps of the doodle.

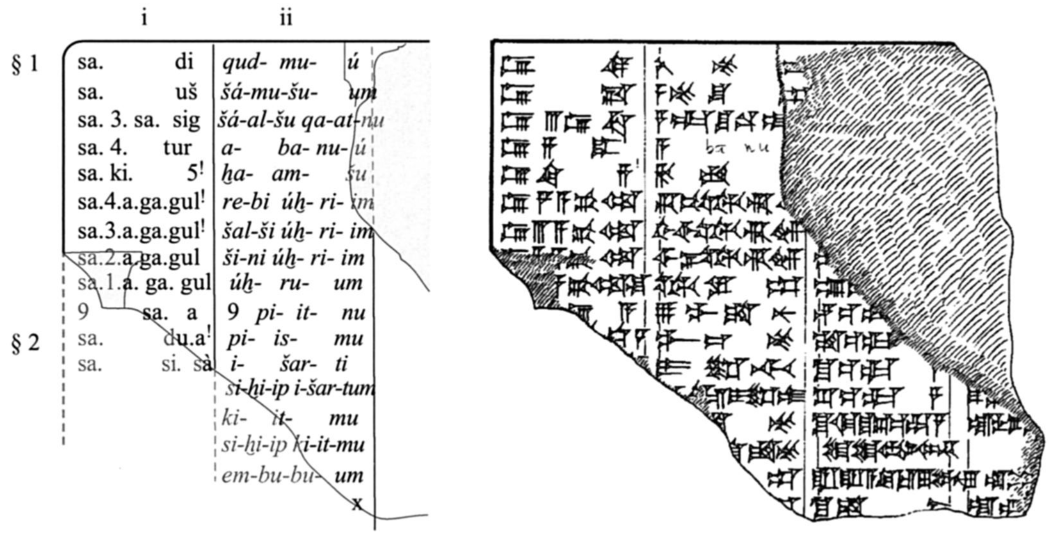


Fig. 10.1. UET VII 126, a fragment from Ur of a copy of the 32nd tablet of the lexical series *Nabnītu*. The copy is published here with the kind permission of the Trustees of the British Museum.

UET VII 126, cols. i-ii

§ 1	sa.di	*qud-mu-ú*	fore string
	sa.uš	*šá-mu-šu-um*	next string
	sa.3.sa.sig	*ša-al-šu qa-a*[*t-nu*]	third, thin string
	sa.4.tur	*a-ba-nu-*[*ú*]	fourth, small string / Ea-created
	sa.ki.5	*ḫa-am-*[*šu*]	fifth string
	sa.4.a.ga.gul	*re-bi úḫ-ri-*[*im*]	fourth rear string
	sa.3.a.ga.gul	*šal-ši úḫ-ri-im*	third rear string
	sa.2.a.ga.gul	*ši-ni úḫ-ri-im*	second rear string
	sa.1.a.ga.gul	*úḫ-ru-um*	rear string
	[9] sa.a	9 *pi-it-nu*	nine strings
§ 2	[sa.d]u.a	*pi-is-mu*	? ? ?
	[sa.si.s]à	*i-šar-ti*	*išartu*

[x x x]	[*si-ḫ*]*i-ip i-šar-tum*	*siḫip išarti*
[x x x]	[*ki-i*]*t-mu*	*kitmu*
[x x x]	[*si-ḫi-ip k*]*i-it-mu*	*siḫip kitmi*
[x x x]	[*em-bu-bu*]*-um*	*embūbu*
....	....	....

The names of 7 of the 9 strings reappear in the Neo-Babylonian table of constants CBS 10996, col. vi (Kilmer, *Or* 29; see section 11 below), together with names for two alternating sets of 7 string pairs (dichords) each.

CBS 10996, obv., col. vi

[1 5	sa	*nīš tuḫ-ri*]	'rise of heel(?)'
[7 5	sa	*še-e-ru*]	'song'
[2 6	sa	*i-šar-tu*$_4$]	'normal'
[1 6	sa	*šal-šá-tu*$_4$]	'third'
[3 7	sa	*em-bu-bu*]	'reed-pipe'
[2 7	sa	4-*tu*]	'4th'
[4 1	sa	šub.murub$_4$]	'fall of middle'
[1 3	sa	giš.šub.ba]	'lot, share'
[5 2	sa	murub$_4$-*tu*$_4$]	'middle'
2 4	[sa	*ti-tur*] murub$_4$-*tu*$_4$	'bridge, middle'
6 3	sa	*kit-mu*	'cover'
3 5!	sa	*ti-tur i-šar-tu*$_4$	'bridge, normal'
7 4	sa	*pi-tu*$_4$	'opening'
4 6	sa	*ṣer-du*	'lament'

sa *qud-mu-ú*	*ù*	sa 5-*šu*	1 5	sa *nīš tuḫ-ri*	*ù* means 'and'
sa 3! *uḫ-ri*	*ù*	sa 5-*šu*	7 5	sa *še-e-ru*	
sa *ša*-ge$_6$	*ù*	sa 4 *uḫ-ri*	2 6	sa *i-šar-tu*$_4$	*ša*-ge$_6$ = *ša-mūšu*
sa *qud-mu-ú*	*ù*	sa 4 *uḫ-ri*	1 6	sa *šal-šá-tu*$_4$	
sa 3-*šú* sig	*ù*	sa 3-*šú uḫ-ri*	3 7!	sa *em-bu-bu*	sig = *qatnu* 'thin'
sa *ša*-ge$_6$	*ù*	sa 3-*šú uḫ-ri*	2 7!	sa 4-*tu*	
sa ^{d}*é-a*.dù	*ù*	sa *qud-mu-ú*	4 1	sa šub.murub$_4$	dù = *banû* 'created'
sa *qud-mu-ú*	*ù*	sa 3-*šú* sig	1 3	sa giš.šub.ba	
sa x 5-*šú*	*ù*	sa *ša*-ge$_6$	5 2	sa murub$_4$-*tu*	murub$_4$-*tu* = *qablītu* 'middle'
sa *ša*-ge$_6$	*ù*	sa ^{d}*é-a*.dù	2 4	sa *ti-tur* murub$_4$-*tu*	

sa 4 *uḫ-ri*	*ù*	sa 3-*šú* sig	6	3	sa [*kit-mu*]
sa 3-*šú* sig	*ù*	sa 5-*šú*	[3	5	sa *ti-tur i-šar-tu*$_4$]
sa 3-*šú uḫ-ri*	*ù*	sa [^{d}*é-a*.dù	7	4	sa *pi-tu*$_4$]
sa ^{d}*é-a*.dù	*ù*	[sa 4 *uḫ-ri*	4	6	sa *ṣer-du*]

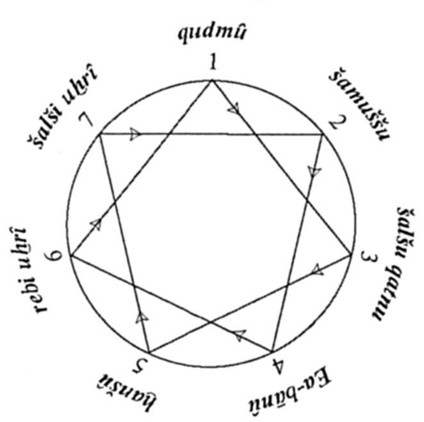
The column (partly reconstructed and corrected above) begins with a brief version of the list of 2 times 7 dichords in terms of only the numbers of the strings, but then, as an afterthought, repeats the list with the dichords expressed also in terms of the full names of the strings.

The names given for the dichords remain to this day largely unexplained. (The reading *tuḫ-ri* 'heel, Achilles tendon?' is due to Mirelman and Krispijn, *Iraq* 71.)

In Fig. 10.2 below, top and bottom, one "primary" set of 7 dichords is identified with the seven sides of a 7/3 star figure, while a "secondary" set of 7 dichords is identified with the seven sides of a 7/2 star figure. Note that while this way of visualizing the two sets of 7 dichords mentioned in col. i of CBS 10996 is *not known* from any Babylonian text, it still is, of course, inspired by the diagram and the table on CBS 1766.

Note also that in Fig. 10.2, below, the sides of the 7/2 star figures are oriented in the same way as in the corresponding diagram in Fig. 6.2 above. In three cases, marked by asterisks, the two numbers defining a dichord correspond to a side in the 7/2 star figure with an opposite direction. This appears to be a mistake made by the author of the text.

primary	
1, 5	***nīš tuḫri***
2, 6	***išartu***
3, 7	***embūbu***
4, 1	***nīd qabli*** (šub.murub)
5, 2	***qablītu*** (murub-*tu*$_4$)
6, 3	***kitmu***
7, 4	***pītu***



secondary	
5, 7*	***šēru***
6, 1*	***šalšatu***
7, 2*	***rebūtu*** (4-*tu*)
1, 3	***isqu*** (giš.šub.ba)
2, 4	***titur qablītu***
3, 5	***titur išartu***
4, 6	***ṣerdu***

Fig. 10.2. A visualization of the two sets of 7 string pairs (dichords) mentioned in CBS 10996.

Note, finally, that in col. vi of CBS 10996, the first set of dichords is ordered *lexicographically* rather than in the order corresponding to successive sides of the 7/3 star figure. More precisely, the 7 dichords in the first set succeed each other in the following way:

1, 5
2, 6 = 1, 5 + 1, 1
3, 7 = 2, 6 + 1, 1
4, 1 = 3, 7 + 1, 1 (8 = 1 mod 7)
5, 2 = 4, 1 + 1, 1
6, 3 = 5, 2 + 1, 1
7, 4 = 6, 3 + 1, 1

Therefore the 7 sides of the 7/3 star figure corresponding to successive *primary* dichords are obtained from each other through repeated rotation by 1/7 of a full revolution.

Similarly, the 7 dichords in the *second* set succeed each other in the following way:

5, 7*
6, 1* = 5, 7 + 1, 1 (8 = 1 mod 7)
7, 2* = 6, 1 + 1, 1
1, 3 = 7, 2 + 1, 1 (8 = 1 mod 7)
2, 4 = 1, 3 + 1, 1
3, 5 = 2, 4 + 1, 1
4, 6 = 3, 5 + 1, 1

Therefore also the 7 sides of the 7/2 star figure corresponding to successive *secondary* dichords are obtained through repeated rotation of the first side by 1/7 of a full revolution.

Note: In Vitale, UF 14, 254, another way of visualizing the two sets of 7 dichords mentioned in col. i of CBS 10996 by use of 7/2 and 7/3 star figures is only superficially related to the method of presentation in Fig. 10.2 above, which is based on the testimony of the text CBS 1766. Vitale was, of course, unaware of the existence of CBS 1766.

A key text for the understanding of Babylonian music theory is UET VII 74+ (Gurney, *Iraq* 56; Dumbrill, *Archaeomusicology*, 48), a small fragment of an Old Babylonian text with two explicit modal retuning algorithms for a string instrument with nine strings (see Fig. 10.3). A second, newly identified fragment of the same text (although not the same clay tablet!) is UET VI/3 899, Mirelman and Krispijn, *Iraq* 71.

The upper half of col i of UET VII 74+ contains what may be § 1 of the text, apparently devoted to some kind of enumeration of string pairs. It is difficult to say precisely how that first paragraph was organized.

In the transliteration below of the text of the fragment, missing parts of the text have been tentatively reconstructed, within straight brackets.

UET VII 74+, col. ii

...	....	....
§ 2.5	[*šum-ma* giš zà.mí *pi-tum*] /	[If the *sammû* (instrument) is *pītu*,]
	[*e-e*]*m-b*[*u-bu-um la za-ku*] /	the *embūbu* is unclear,
	ša-al-š[*a-am qa-at-na-am ta-na-sá-aḫ-ma*] /	the third, thin, you shall tighten, then
	e-em-bu-bu-[*um iz-za-ku*] /	the *embūbu* will be clear.
§ 2.6	*šum-ma* giš*z*[à.mí *e-em-bu-bu-um*] /	If the *sammû* is *embūbu*,
	ki-it-mu-um [*la za-ku*] /	the *kitmu* is unclear,
	re-bé úḫ-ri-im [*ta-na-sà-aḫ-ma*] /	the fourth rear you shall tighten, then
	ki-it-mu-um iz-[*za-ku*] /	the *kitmu* will be clear.
§ 2.7	*šum-ma* giš zà.mí *k*[*i-it-mu-um*] /	If the *sammû* is *kitmu*,
	i-šar-tum la za-[*ka-at*] /	the *išartu* is unclear,
	ša-mu-ša-am ù úḫ-ri-a-a[*m ta-na-sà-aḫ-ma*] /	the *šamuššu* and the rear you shall tighten,
	i-šar-tum iz-za-[*ku*] /	then the *išartu* will be clear.
end	*nu-su-*[*ḫu-um*] /	Tightening.
	šum-ma giš zà.mí *i-šar-t*[*um*] /	If the *sammû* is *išartu*,
§ 3.1	*qá-ab-li-ta-am* <*la za-ku-ta-am*> *ta-al-pu-*[*ut*] /	the *qablītu* <unclear> you played,
	[*š*]*a-mu-ša-am ù úḫ-ri-a-am te-*[*né-e-am-ma*] /	the *šamuššu* and the rear you shall loosen,
	[giš z]à.mí *ki-it-mu-*[*um*] /	then the *sammû* will be *kitmu*.
§ 3.2	[*šum*]*-ma* giš zà.mí *ki-it-m*[*u-um*] /	If the *sammû* is *kitmu*,
	[*i-ša*]*r-ta-am la za-ku-ta-am t*[*a-al-pu-ut*] /	the *išartu* unclear you will play, then
	[*re-bi*] *úḫ-ri-im te-né-e-*[*am-ma*] /	the fourth rear you shall loosen, then
	[giš zà.mí *e-em-bu-bu-um*] /	the *sammû* will be *embūbu*.
	....	....

By luck, the fragment UET VII 74+ contains both the end of one modal retuning algorithm and the beginning of another. (A (recursive) *algorithm* is a procedure in several steps, where each step is structurally similar to the previous step.)

The meaning of the preserved beginning of the *second modal retuning algorithm* can be vaguely explained as follows:

§ 3.1. If the string instrument is tuned to the *išartu* mode, and the *qablītu* dichord is dissonant ('unclear'), loosen the *šamuššu* (= second) string and the rear (= ninth) string. The string instrument becomes tuned to the *kitmu* mode.

§ 3.2. If the string instrument is tuned to the *kitmu* mode, and the *išartu* dichord is dissonant, loosen the fourth rear (= the sixth) string. The string instrument becomes tuned to the *embūbu* mode.

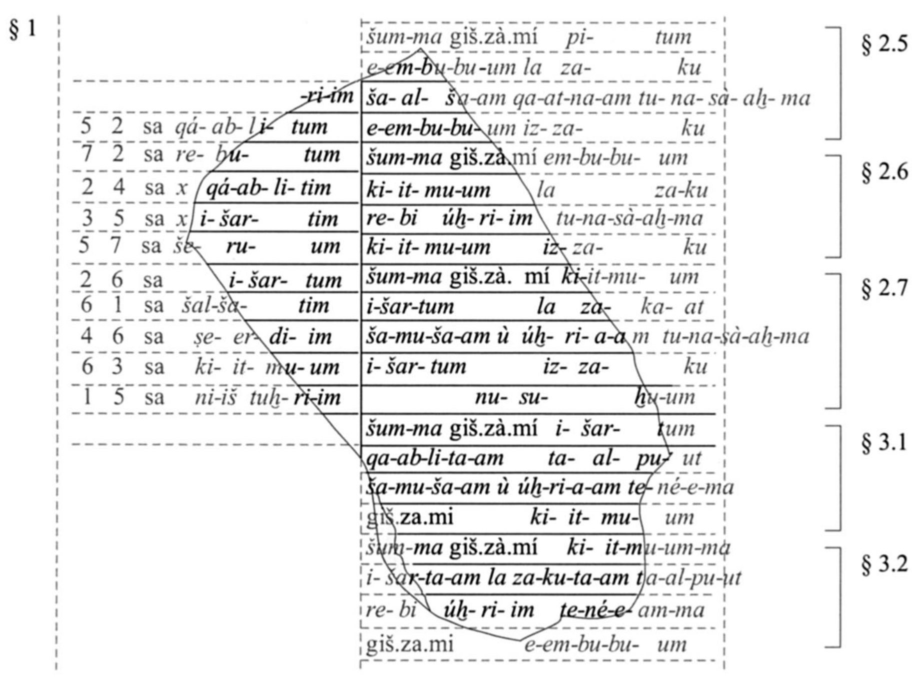


Fig. 10.3. UET VII 74, a fragment of an OB text from Ur with two retuning algorithms.

Evidently, as shown in Fig. 10.4 below, the modes in this retuning algorithm (after the obvious reconstruction) follow each other in the same order as the sides in the 7/3 star figure.

The application of the modal retuning algorithm in UET VII 74+, § 3 presupposes that the string instrument has been tuned already to some mode. The tuning to the *initial generative mode*, clearly the *išartu* or 'normal' mode, is easily obtained through application of the following *initial tuning algorithm*: After string 2 has been tuned in some arbitrary but appropriate way, string 6 is tuned to make the *išartu* dichord 2, 6 'clear' (in modern terms an ascending *fifth*). Next, string 3 is tuned to make the *kitmu* dichord 6, 3 'clear' (a descending *fourth*). In the third step, string 7 is tuned to make the *embūbu* dichord 3, 7 'clear' (an ascend-

2, 6	***išartu***
6, 3	***kitmu***
3, 7	***embūbu***
7, 4	***pītu***
4, 1	***nīd qabli***
1, 5	***niš tuḫri***
5, 2	***qablītu***

Fig. 10.4. Dichords following each other as the sides of the 7/3 star figure.

ing *fifth*). Then string 4 is tuned to make the *pītu* dichord 7, 4 'clear' (a descending *fourth*), string 1 is tuned to make the *nīd qabli* dichord 4, 1 'clear' (a descending *fourth*), and string 5 is tuned to make the *niš tuḫri* dichord 1, 5 'clear' (an ascending *fifth*). The inevitable end result of this straightforward initial tuning algorithm is that the *qablītu* dichord 5, 2 becomes 'unclear' (in modern terms a disharmonic *tritone*, or more precisely, *an augmented fourth*, alternatively *a diminished fifth*) and cannot be made 'clear' without disturbing the given initial tuning of string 2.

Both this assumed *initial tuning algorithm* and the *modal retuning algorithm* in § 3 can be explained with reference to the 7/3 star figure in Fig. 10.4. As shown in the first diagram in Fig. 10.5 below, in the generative *išartu* 'normal' mode all the dichords corresponding to the sides of the 7/3 star figure are 'clear', except the *qablītu* dichord 5, 2.

In the first step of the retuning algorithm in § 3, string 2 is 'loosened' so that the *qablītu* dichord 5, 2 becomes 'clear.' As a result, the *išartu* dichord 2, 6 becomes 'unclear,' and the *kitmu* dichord 6, 3 becomes the initial dichord of this new mode, therefore called the *kitmu* mode. See the second diagram in Fig. 10.6. And so on.

After six steps of this modal retuning algorithm in § 3 of UET VII 74+, all strings except string 5 have been 'loosened.' The initial interval of this mode is the *qablītu* dichord, and the *niš tuḫri* dichord 1, 5 is 'unclear.' Therefore, this is called the *qablītu* mode. See the seventh diagram in Fig. 10.5. In the last step of the retuning algorithm, finally, string 5 is 'loosened' as well. Now the *qablītu* dichord is 'unclear' again, and the configuration is the same as in the initial *išartu* mode, only with *all* strings 'loosened.'

It is not only the retuning algorithm for seven modes in UET VII 74+, § 3 that can be explained easily in terms of the 7/3 star figure in Fig. 10.4. Also the varying distributions of *tones* and *semitones* in the seven modes can be explained without trouble by reference to the 7/3 star figure (although there is no known document indicating that the authors of the Babylonian texts discussed in the present paper were aware of this possibility).

Consider, for instance, the 7/3 star figure for the *išartu* mode in Fig. 10.5. In that star figure, the dichord 2, 3 can be construed as a combination of the dichord 2, 6 (*a descending fifth*) and the dichord 6, 3 (*an ascending fourth*). Therefore, the dichord 2, 3 can be understood as a *regular second* (or rather a *major second*, in modern notation a *tone*). In the *išartu* mode, other regular seconds (or tones) of the same kind are 3, 4 and 4, 5, as well as 6, 7 and 7, 1. The situation is different in the case of the dichord 5, 6, which can be construed as a combination of the dichord 5, 2 (*an augmented ascending fourth*) and the dichord 2, 6 (a *descending fifth*). Therefore, the dichord 5, 6 can be called a *minor second* (a *semitone*). Similarly, the dichord 1, 2 can be understood as a combination of the dichord 1, 5 (a *descending fifth*) and the dichord 5, 2 (*an augmented ascending fourth*). Consequently, 5, 2 is another minor second (or semitone).

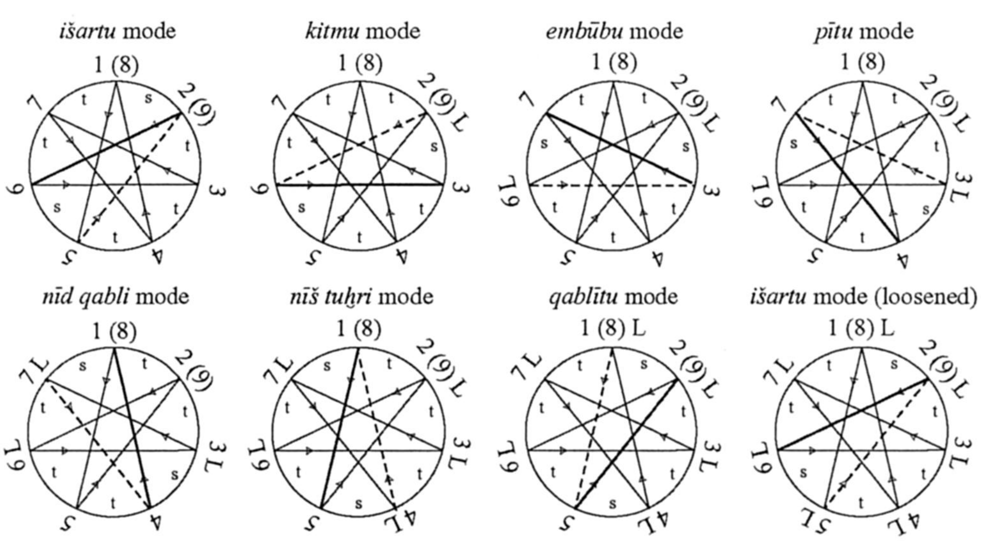


Fig. 10.5. The retuning algorithm in UET VII 74+, § 3 in terms of the 7/3 star diagram. (L = loosened.)

In the same way, it can be seen that in all the 7/3 star figures for the seven modes in Fig. 10.4, the 'unclear' dichord (dashed) is next to two minor seconds (semitones), while all the other seconds are major seconds (tones). In Fig. 10.5, the semitones are indicated by the letter *s*, while the tones are indicated by the letter *t*.

In modern notations, the retuning algorithm in Fig. 10.5 can be expressed as follows:

It is interesting that the result of applying the (partly hypothetical) Old Babylonian retuning algorithms based on sequences of descending fifths

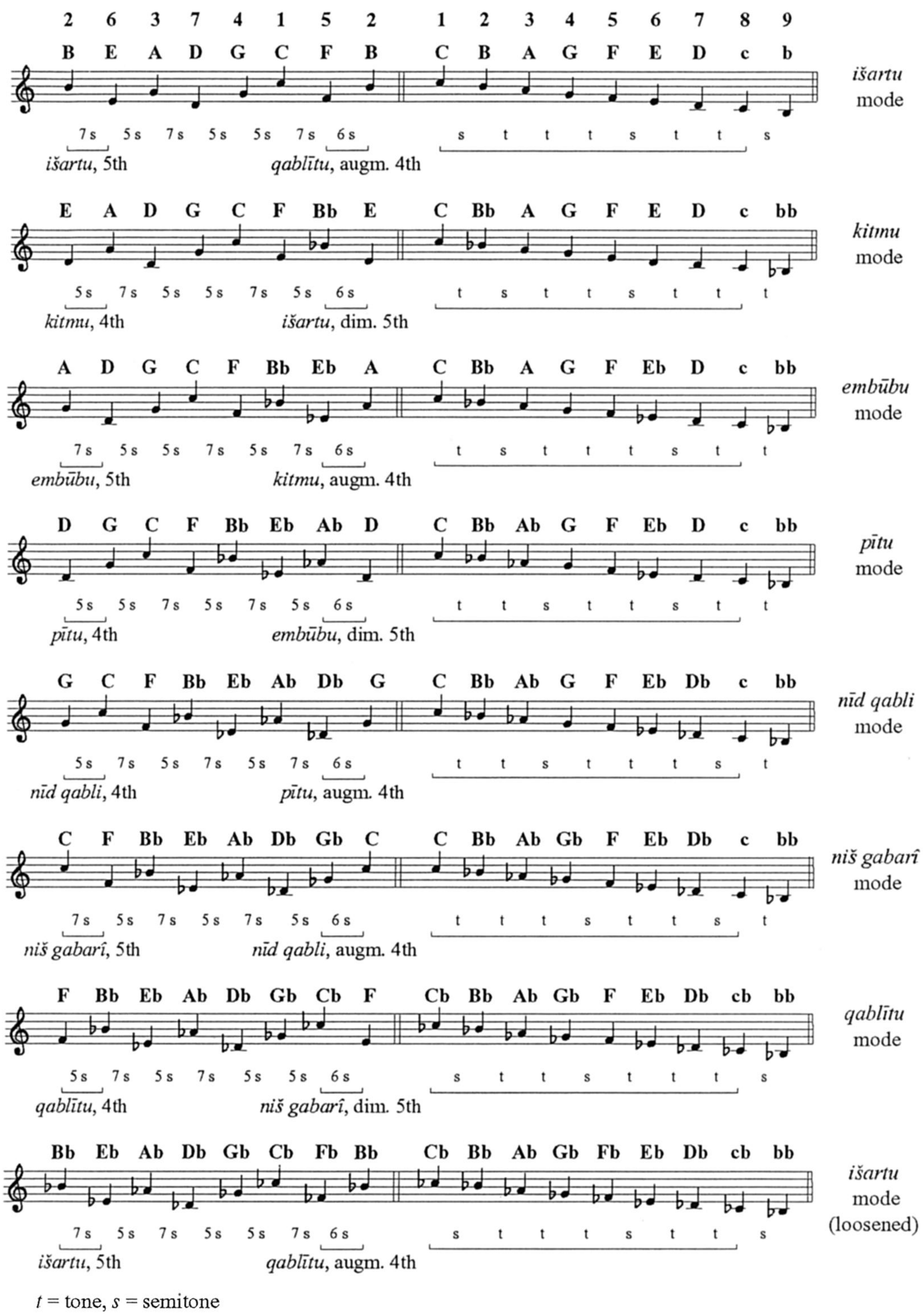


Fig. 10.6. A modern (anachronistic) interpretation of the OB retuning algorithm in UET VII 74+, § 3.

initial tuning algorithm is used a second time in order to obtain the chosen mode. The *pītu* mode, for instance, can be obtained with departure from string 7, tuning string 4 so that the *pītu* dichord becomes clear, then tuning string 1 so that the *nīd qabli* dichord becomes clear, and so on. The way to proceed is shown clearly by the 7/3 star diagram in Fig. 10.4.

Now, consider instead the meaning of the preserved end of the *first modal retuning algorithm*, the one of UET VII 74+, § 2, which can be explained vaguely as follows:

§ 2.5. If the string instrument is tuned to the *pītu* mode, and the *embūbu* dichord is dissonant ('unclear'), tighten the third string. The *embūbu* dichord becomes consonant ('clear').

§ 2.6. If the string instrument is tuned to the *embūbu* mode, and the *kitmu* dichord is dissonant, tighten the fourth rear (= the sixth) string. The *kitmu* dichord becomes consonant.

§ 2.7. If the string instrument is tuned to the *kitmu* mode, and the *išartu* dichord is dissonant, tighten the *šamuššu* (= second) and the rear (= ninth) string. The *išartu* dichord becomes consonant.

Tuning by tightening.

and ascending fourths, as illustrated by the 7/3 star figure in Fig. 10.4, will automatically lead to what in modern terminology may be called *7 different descending diatonic heptatonic modes*.

Note that all the modes in Fig. 10.5 can be obtained also *without* the use of the modal retuning algorithm, namely as follows: First the initial tuning algorithm is used in order to obtain the *išartu* mode. Then the

This first modal retuning algorithm, too, can be explained in terms of the 7/3 star diagram, as in Fig. 10.7 below.

All the modes in Fig. 10.7 can be obtained as follows, *without* the use of the modal retuning algorithm: first the initial tuning algorithm is used in order to obtain the *išartu* mode. Then the initial tuning algorithm is used a second time in order to obtain the

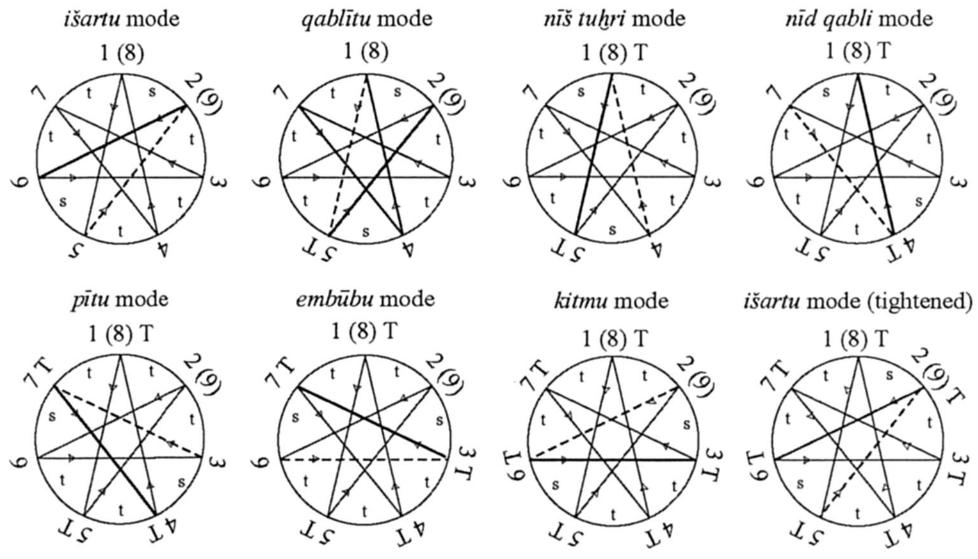


Fig. 10.7. The retuning algorithm in UET VII 74+, § 2, in terms of the 7/3 star diagram. (T = tightened.)

chosen mode. The *kitmu* mode, for instance, can be obtained with departure from string 6, by first tightening string 6, then tuning string 3 so that the *kitmu* dichord becomes clear, tuning string 7 so that the *embūbu* dichord becomes clear, and so on. The way to proceed is again shown clearly by the 7/3 star diagram in Fig. 10.4.

In modern notations, the retuning algorithm of UET VII 74+, § 2 can be expressed as:

The ordering of the dichords in the order of the sides of the 7/3 star figure, beginning with the *išartu* dichord, seems to have been the prevailing standard. At least, this is what is suggested by the Assur text (VAT 10101) a long catalog of vocal and instrumental music, where in particular (see Kilmer, *Festschrift Landsberger*, 267) a list of love songs is summarized in the following way (ll. 45-52):

23 *irātu ša e-šir-te akkadî*[ki]	23 love songs in the *išartu* mode, Akkadian
17 *irātu ša ki-it-me*	17 love songs in the *kitmu* mode
24 *irātu ša eb-bu-be*	24 love songs in the *embūbu* mode
4 *irātu ša pi-i-te*	4 love songs in the *pītu* mode
[...] *irātu ša ni-id* murub$_4$	[...] love songs in the *nīd qabli* mode
[...] *irātu ša ni-iš tuḫ-ri*	[...] love songs in the *nīš tuḫrî* mode
[...] *irātu ša* murub$_4$*-te*	[...] love songs in the *qablītu* mode
[...] *akkadî*[ki]	[Total ... love songs], Akkadian

The same ordering of the dichords can be observed in column ii of the table on CBS 1766 (see Fig. 6.1 above).

CBS 1766. *A Clue to the Provenance and the Date of the Clay Tablet*

It seems to be clear now that CBS 1766 is a text with a mixed topic. On one hand, there is the geometric topic of three kinds of 7-sided figures, both the 7/3 star figure which is explicitly depicted, the '7-side' (regular heptagon) whose sides are parallel to the sides of the 7/3 star figure, and the 7/2 star figure whose sides are also parallel with the sides of the 7-side. Indeed, according to the interpretation suggested in Fig. 6.2 above, the text of the partially preserved headings above columns ii-iv seems to refer to two of these three kinds of 7-sided figures. Regrettably, the text of the corresponding heading above columns v-vi is not preserved.

On the other hand, CBS 1766 also concerns the topic of Old Babylonian music theory, made obvious through the labelling of the seven points of the star figure by the names of seven strings of the *sammû*, and through the listing in column ii of the seven dichords in the order of the sides of the 7/3 star figure.

In this connection, it is potentially important that traces are preserved also of the inscription in column xi, the last column of the table on CBS 1766, close to the right edge.[3] The heading over that column appears to be mu.[bi.im] 'its name,' while traces of the inscriptions in lines 1 and 2 of the same column can be read as *i-[šar-tum]* and *k[i-it-mu-um]*, the names of the dichords in lines 1-2 of col. ii.

If the suggested readings of the preserved traces of inscriptions in the last column on CBS 1766 are correct, that means that the table on CBS 1766 in a certain sense is a close parallel to the table on the famous Old Babylonian mathematical table text Plimpton 322 (Friberg, MSCT 1, App. 8). This observation, in its turn, is important because it means that conclusions can be drawn about both the date and the provenance of CBS 1766.

Indeed, on p. 33 of an interesting paper about "tables and tabular formatting" in cuneiform texts (in Campbell-Kelly *et al.*, *The History of Mathematical Tables*), Robson writes that

> "... there is only one known mathematical cuneiform tablet which is conspicuously indebted to administrative practise. Plimpton 322 has achieved such an iconic status as the Mesopotamian mathematical tablet *par excellence* that it comes as quite a shock to

[3]) Collated by G. Frame, personal communication.

Fig. 10. 8. A modern (anachronistic) interpretation of the OB retuning algorithm in UET VII 74+, § 2.

realize how odd it is. Its fame derives from its mathematical content: fifteen rows of four extant columns containing sophisticated data relating to Pythagoras' theorem. The fact that this data is laid out in a landscape-oriented headed table, with a final heading MU.BI.IM ('its name') for the non-numerical data, has gone completely unremarked. These, of course, are formal features of administrative tables from Larsa during the period of rigorous standardization in the 1790-80s BCE."

Just like on Plimpton 322, the data on CBS 1766 are laid out in a landscape-oriented table with headings, in particular with a final heading mu.bi.im for the non-numerical data. Therefore, the conclusion must be that CBS 1766, like Plimpton 322, in all probability is an *Old Babylonian text from Larsa*, dating to the period 1790-1780 BC.

UET VII 74+ *in the Context of Old Babylonian "Rational Practice Texts"*

In a section of Ritter's interesting paper in Chemla, *History of Science*, 177-200, a typical Old Babylonian mathematical text is considered in the context of what is called there "rational practice texts." Ritter examines the grammatical structure of the text, and specifically the verbal chains. What he finds is that in this (and most other) *Old Babylonian mathematical texts*, the verbs in the *statement* of the problem are in the *preterite* (roughly corresponding to the English past tense), while the verbs in the *solution procedure* are in the *durative* (roughly corresponding to the English future tense), or in the *imperative*.

Ritter further states that this kind of rigidity of syntax can be observed in only three other genres of Old Babylonian texts, those of *divination*, *medicine*, and *jurisprudence*. In one kind of *divination* text, for instance, the *form that oil takes* when poured on water is described with verbs in either the *preterite* or the *stative* (describing a constant state), while the corresponding *prediction* is expressed with verbs in the *durative* or the *stative*. In a cited example, the text says

If, from the middle of the oil, two drops *came out* and one *was* large and the other small,
the man's wife *will give* birth to a boy; for the sick man: he *will recover*.

In the case of *medical procedure texts*, the *presentation of the medical problem* is expressed with verbs in the *preterite* or the *stative*, while the *medical solution* to the problem is expressed with verbs in the *durative*. In a cited example, the text says

If a man was *stung* by a scorpion,
you *will apply* 'ox excrement' and he *will recover*.

In *juridical procedure texts*, finally, the *presentation of the case* is expressed in terms of verbs in the *preterite*, followed by a verb in the *perfect* (English present perfect), while in the *solution to the case* the verbs are in the *durative*. In a cited example, the text says

If a man *accused* a(nother) man and *charged* murder (against him), but he *was not convicted*,
his accuser *will be killed*.

Note that a conspicuous (and typical) common feature of the three cited examples is that they all start with the word 'if' (Akk. *šumma*). Old Babylonian mathematical texts, on the other hand, rarely start this way, although they do in a few instances, such as the geometric algorithm text VAT 8393 (Friberg, *Amazing Traces*, 434), and the Eshnunna texts IM 52301 (Høyrup, *Lengths*, *Widths*, *Surfaces*, 213) and IM 67118 (= Db_2-146) (*ibid.*, 257).

This conspicuous feature is shared also by each paragraph in the retuning algorithm text UET VII 74+, §§ 2-3. Moreover, in each one of those paragraphs, the *statement of the problem* with the given tuning of the *sammû* instrument is expressed with verbs in the *stative*, while the *solution to the problem* is expressed in terms of verbs in the *durative*. Therefore, it appears that *the retuning algorithm text UET VII* 74+, §§ 2-3 belongs to the same category of "*rational practice texts*" as Old Babylonian mathematical procedure texts, divination texts, medical procedure texts, and juridical procedure texts!

11. *CBS* 10996. *A Neo-Babylonian(?) Table of Constants*

Photos of obverse and reverse of CBS 10996, a large fragment of a Neo-Babylonian(?) table of constants, were published by Kilmer in *Or* 29, together with a translation of the text and a commentary. The new copies of the text in Figs. 11.1 and 11.4 were kindly made for the author by F. Al-Rawi.

The text on one side of the fragment is almost perfectly preserved, while very little remains of the text on the other side. Although the imperfect state of preservation of the clay tablet makes it difficult to be absolutely sure, apparently the well preserved side of the tablet is the reverse.[4]

Only 15 lines of the first column on the obverse are partially preserved, but the appearance of, for instance, the terms sag 'front, short side,' uš 'length, long side,' dal 'transversal,' and gúr 'curve, circle' in this brief list makes it obvious that this is what remains of a table of constants with parameters for simple plane geometric figures. In particular, the lines (i 9'-11')

5 gúr	5 (for the) circle
1 bal [gúr]	1 (for the) ratio [of a circle]
20' dal [gúr]	20 (for the) transversal [of a circle]

refer to the following well known Babylonian rules for the area A and diameter d of a circle:

$A = 5\ (\cdot\ 1/60) = 1/12$ for a circle with a circumference of unit length

$d = 20\ (\cdot\ 1/60) = 1/3$ for a circle with a circumference of unit length.

More specifically, this means that (approximately)

$A = 5\ (\cdot\ 1/60) \cdot$ square of a for a circle with the circumference a

$d = 20\ (\cdot\ 1/60) = 1/3 \cdot a$ for a circle with the circumference a.

The mention of the 'ratio' 1 is without known precedent. Presumably, it refers to the ratio $a/1$ which, of course, is equal to 1 in a circle with a circumference of unit length.

As shown in Figs. 11.2-4 below, the text on the *reverse* of CBS 10996 contains fairly well preserved parts of three columns of text, presumably columns iv-vi. The general layout of the text is shown in the outline of the fragment below.

The table of constants on CBS 10996 has a very inhomogeneous, mixed content, just like a number of other known Old Babylonian tables of constants (see Friberg, in *Changing Views*, 64-67; Robson, *Mesopotamian Mathematics*, xiii, 193-207). The explanation is probably that there existed no "canonical" table of constants. Instead, tables of constants may typically have been produced by teachers of mathematics who sporadically made notes, for future use in the classroom, of numerical data that they found in mathematical (and other) texts that happened to be available to them. Furthermore, entries in Babylonian tables of constants are usually so brief that it is impossible to understand what they refer to, except in the lucky cases when texts are known where the constants appear in a comprehensible context.

Thus, for instance, the first preserved section in col. iv on the reverse of CBS 10996 contains a list of constants in some way related to heaps of še.giš.i 'sesame.' Since no text, or at least no mathematical text, is known where such constants appear in a natural way, there is no obvious explanation for these sesame constants.[5]

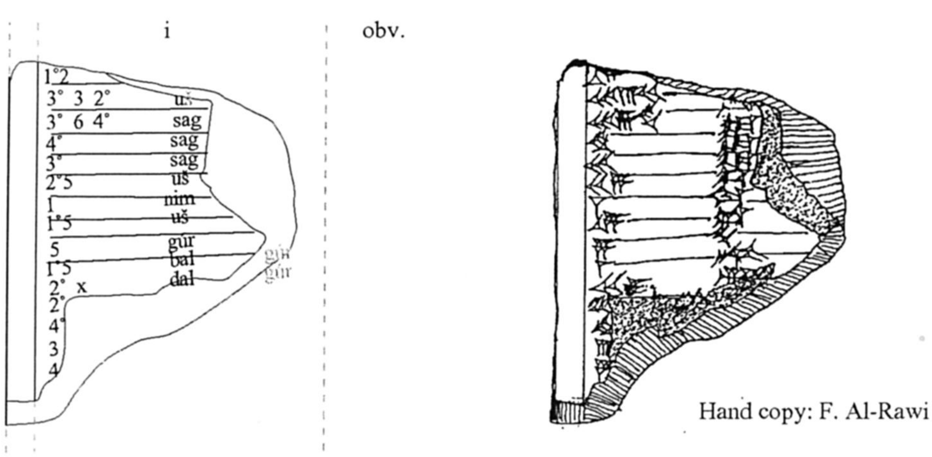


Fig. 11.1. CBS 10996, obv. Conform transliteration and copy. Copy: F. Al-Rawi.

[4]) Collated by G. Frame, personal communication.

[5]) Some Neo-Assyrian tablets from an archive in the South Palace of Nebuchadnezzar II in Babylon contain accounts mentioning both sesame and sesame oil, with an exchange rate of 1 unit of oil for 6 or 7 units of sesame (O. Pedersén, *Studia Orientalia* 106,

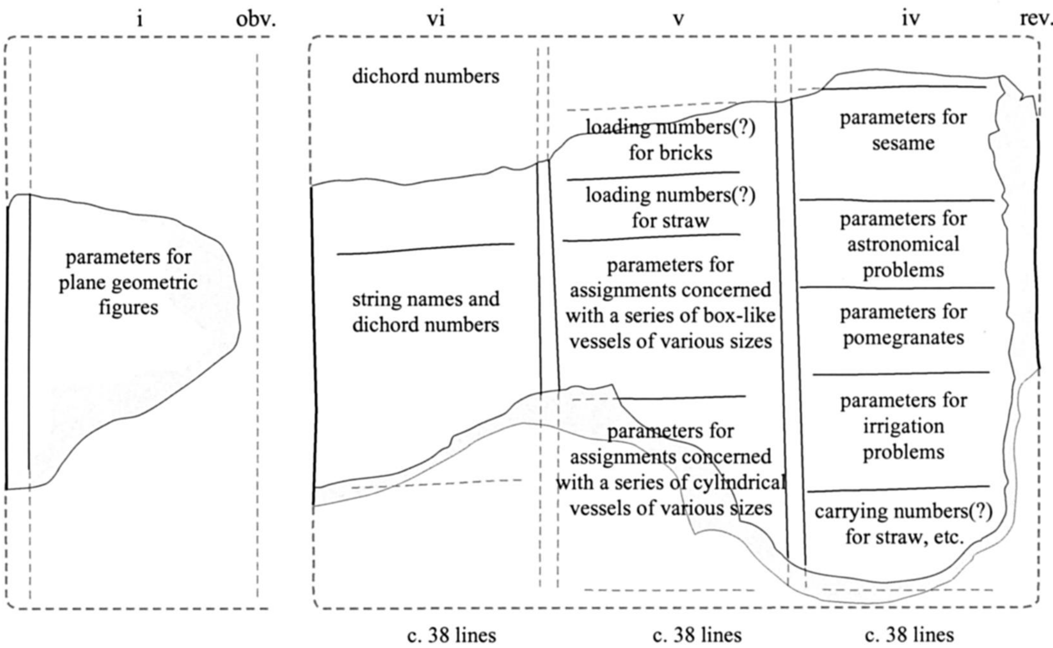


Fig. 11.2. CBS 10996. An outline of the fragment, with an indication of the general layout of the text.

The second section in col. iv, on the other hand, contains constants well known from astronomical texts, such as the astronomical and astrological compendium *Enūma Anu Ellil* (see Friberg *et al.*, *BaM* 21, 496-499 and Robson, *Mesopotamian Mathematics*, Sec. 8.2). In particular, mú and šú ša [d]30 mean the 'first rising' and the 'first setting' of the moon ('the god 30'), igi.du$_8$.a means 'visibility' (of the moon), while *u$_4$-mu* and ge$_6$ mean 'day' and 'night.'

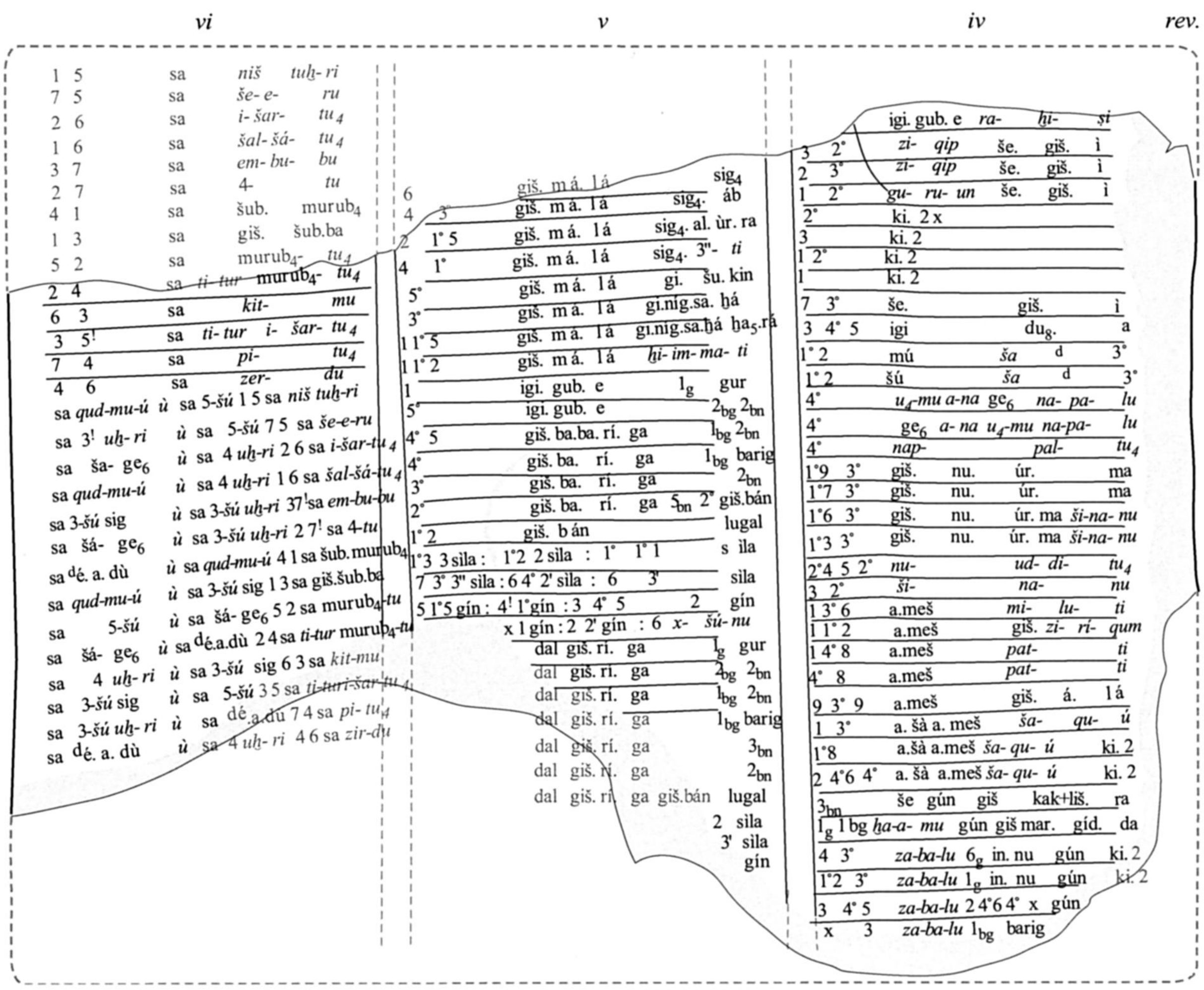


Fig. 11.3. CBS 10996. A Neo-Babylonian table of constants of mixed content. Conform transliteration.

198. The constants for sesame mentioned in CBS 10996 are of a different nature. See also Bongenaar, *Ebabbar*, with a discussion of the oil pressers on pp. 261-287. Various rates are mentioned in fn. 241 on p. 266.

Fig. 11.4. CBS 10996. A Neo-Babylonian table of constants of mixed content. Copy F. Al-Rawi.

The third section in col. iv, is concerned with constants for giš.nu.úr.ma 'pomegranates.' In this case, too, the term does not appear in any known mathematical texts, so the meaning of the constants remains unknown.

The fourth section is concerned with constants for irrigation (a.meš means 'water'). Several of these constants are known from Old Babylonian mathematical texts (see Robson, *Mesopotamian Mathematics*, Sec. 6.5).

The fifth and final section in col. iv contains constants in some way associated with problems concerning transportation of commodities measured in capacity measure. In particular, gún giš.mar.gíd.da means 'load of a wagon,' and *zabālu* means 'to carry.'

In col. v, the first preserved section mentions constants called giš.má.lá 'cargo-boat,' meaning either "molding numbers" or "loading numbers," for four kinds of bricks, namely sig_4 (ordinary rectangular bricks, measuring 1/2 cubit × 1/3 cubit × 5 fingers), sig_4.áb (half-bricks, 2/3 cubit × 1/3 cubit × 5 fingers), sig_4.al.ùr.ra (square bricks, 2/3 cubit × 2/3 cubit × 5 fingers), or sig_4.2/3-*ti* (larger rectangular bricks, 18 fingers × 12 fingers × 5 fingers). Such constants are well known from various Old Babylonian problem texts and tables of constants (see Friberg, in *Changing Views*, Sec. 4.1; MSCT 1, Sec. 7.3).

The second section in col. v is structured in a similar way, with constants called giš.má.lá, only this time not for bricks but for reeds and reed bundles. The exact meaning of these reed constants is not known.

On CBS 10996, the table of constants in the proper sense ends with the section of reed constants. What then follows, in the lower part of col. v, is not a table of constants but a seemingly peculiar enumeration of sexagesimal numbers and capacity measures, making no real sense in a table of constants. It is rather in several ways similar to the table of parameters for a series of mathematical problems concerned with measuring vessels in the Old Babylonian theme text YBC 4669. (See Friberg, MSCT 1, Sec. 4.7, in particular Fig. 4.7.) It is likely, therefore (see the clarifying discussion in Friberg, *BaM* 28, 309) that the numbers and capacity measures tabulated in the lower part of col. v on CBS 10996 are the data for two series of exercises much like the ones in YBC 4669, one series for box-like measuring vessels, and a second series for cylindrical measuring vessels.

In view of the proposed explanation of mathematical tables of constants as a mathematics teachers' work notes for future use in the class room, there is nothing strange in the inclusion in this text of data for a couple of series of mathematical exercises.

Similarly, there is nothing strange in the inclusion of the table of names for fourteen dichords in the upper part of col. vi of CBS 10996, actually at the very end of the text, after the author of the text had run out of mathematical constants that he wanted to make notes of. Also, it is not strange that when he saw that there was still some space available in the lower part of col. v, he decided to be more explicit, calling by name not only the fourteen dichords but also the seven strings in terms of which the dichords were defined.

It is more surprising that the primary series of dichords is not recorded in the same order as the sides of the 7/3 star figure (2, 6; 6, 3; *etc.*; see Fig. 10.4), but rather in lexicographic order (1, 5; 2, 6; *etc.*). The reason may be that the author of the text did not really understand the tuning procedure based on the sequence 2, 6; 6, 3; *etc.*

The order of the secondary series of dichords (7, 5; 1, 6; *etc.*) seems to be coupled (in a somewhat unorganized way) to the order of the primary series of dichords. The purpose of this secondary series of dichords is not well understood. It has been proposed, in Smith and Kilmer, SMA I, that the secondary series of dichords was used for "fine tuning," but according to Dumbrill (*Archaeomusicology*) the proposal is unrealistic. So, maybe, this secondary series was contrived, in a purely theoretical way, as an attempt to give a musical meaning to the dichords corresponding to the sides of a 7/2 star figure, as suggested by the (incomplete) evidence of CBS 1766 (Fig. 10.2).

It is remarkable that Babylonian music theory seems to have been closely connected with Babylonian mathematics. This is shown not only by CBS 10996, where the names of the 14 dichords are recorded in a mathematical table of constants, but also by CBS 1766, where three kinds of 7-pointed star figures (a regular heptagon, a 7/2 star figure, and a 7/3 star figure) apparently are considered both as geometric objects and as a visualization of the 14 dichords. Last, but not least, the retuning algorithm in UET VII 74+, col. ii is both in form and in context very much reminiscent of a mathematical recursive algorithm. (Cf., for instance, the ascending and descending geometric recursive algorithms in VAT 8393, Friberg, *Amazing Traces*, App. 1, used for the construction of a chain of trapezoids with fixed diagonals.)

Note: There seem to have been about 38 lines in each column on the reverse of CBS 10996, and it is warranted to assume that also the three (or four?) columns on the destroyed obverse contained about 38 lines each. Since the table of constants in the proper sense ends in the middle of column v, a reasonable estimate is that CBS 10996 originally mentioned, at least, something like $4 \cdot 38 + 14 = 166$ constants. This makes CBS 10996 (in its original, intact form) by far the most extensive of all known Babylonian tables of mathematical or technical constants. (Compare with, for instance, TMS 3 with 70 entries and YBC 5022, Neugebauer und Sachs, MCT text Ud, with 66 entries. The badly preserved text G = IM 49949, admittedly contains a large number of entries, but most of those entries list mathematical problem types, not constants.) Against this background, it is most unfortunate that so much of the obverse of CBS 10996 is lost, but so much more fortunate that column vi with its musical terms is so well preserved!

12. *On Greek "Pythagorean" Music Theory and Ratios of String Lengths*

It would be extremely difficult to try to give a brief and comprehensible account of ancient Greek music theory in general, for a number of reasons. The account below will be concentrated to a narrow and limited discussion of only sources for what is known about Greek music theory in the particular cases when it is concerned with *diatonic heptatonic scales of the Babylonian type*, mainly expressed in terms of ratios of string lengths.

Pythagoras as the Alleged Discoverer of Epimoric String Ratios

The discovery that musical consonance is directly related to numerically simple ratios of string lengths was attributed to Pythagoras himself by his followers, the so called Pythagoreans. How the discovery allegedly was made is described in a well known, but certainly both historically and physically incorrect anecdote, which begins as follows:

> **Nicomachus**, ***Enchiridion***, **Ch. 6** (the beginning of the 2nd century BC; Barker, GMW II, 256-258)
> "... Happening by some heaven-sent chance to walk by a black-smith's workshop, he (Pythagoras) heard the hammers beating iron on the anvil and giving out sounds fully concordant in combination with one another ... and he recognized among them the consonance of the octave and those of the fifth and the fourth. He noticed that what lay in between the fourth and the fifth was itself discordant, but was essential in filling out the greater of these intervals ..."

Pythagorean String Ratios in Plato's Timaeus

One of the oldest known references to ratios of string lengths is contained (implicitly) in a famous passage in Plato's dialogue *Timaeus*, which describes how the divine 'Craftsman' began his creation of the Soul of the Universe by dividing a mixture of the Same, the Different, and the Being into a number of components in the following way:

> **Plato**, ***Timaeus*** **[35b-36b]** (the first half of the 4th century BC; Barker, GMW II, 59-60)
> "... This is how he began to divide. First he took away one part from the whole; then another, double the size

of the first, then a third, *hemiolic* with respect to the second and triple the first, then a fourth, double the second, then a fifth, three times the third, then a sixth, eight times the first, then a seventh, twenty-seven times the first."

"Next he filled out the double and triple intervals, once again cutting off parts from the material and placing them in the intervening gaps, so that in each interval there were two means, the one exceeding and exceeded by the same part of the extremes themselves, the other exceeding and exceeded by an equal number. From these links within the previous intervals there arose *hemiolic*, *epitritic* and *epogdoic* intervals; and he filled up all the epitritics with the epogdoic kind of interval, leaving a part of each of them, where the interval of the remaining part had as its boundaries, number to number, 256 to 243. And in this way he had now used up all the mixture from which he cut these portions."

Strange terms in this passage, borrowed from Pythagorean music theory, are 'hemiolic,' from Greek *hemiólios* ('a half and a whole,' meaning 1 1/2), 'epitritic,' from Greek *epítritos* ('a third more,' meaning 1 1/3), and 'epogdoic,' from Greek *epógdoos* ('an eighth more,' meaning 1 1/8). These three expressions are all 'epimoric,' from Greek *epimórios* ('a part more,' meaning 1 $1/n$, for some small integer n). Incidentally, such expressions are not unusual in Old Babylonian mathematical texts, where they typically occur as coefficients in quadratic equations. See, for instance, § 5 of the Old Babylonian mathematical catalog text BM 80209 (Friberg, *Amazing Traces*, 29).

Note that commonly used translations of the kind $(n + 1)/n$ for 'epimoric,' and 3/2, 4/3, 9/8 for 'hemiolic,' 'epitritic,' 'epogdoic,' are somewhat anachronistic. Indeed, the earliest documented use of common fractions occurs (implicitly) in the Egyptian demotic mathematical papyrus P.BM 10520 § 5 (early(?) Roman). (See Friberg *Unexpected Links*, 150-155.) Thus, when Plato awkwardly writes "the remaining part had as its boundaries, number to number, 256 to 243," that is precisely because he can only express as a ratio (less anachronistically, 256 : 243) what we would write simply as the common fraction 256/243.

In the first of the cited paragraphs from the *Timaeus*, Plato divides the divine mixture into parts of the relative sizes

a, $2\ a$, $1\ 1/2 \cdot 2a$, $2 \cdot 2a$, $3 \cdot (1\ 1/2 \cdot 2\ a)$, $8\ a$, $27\ a$.

These relative sizes can also be expressed as

$1\ a$, $2\ a$, $3\ a$, $4\ a$, $9\ a$, $8\ a$, $27\ a$,

where 4 and 9, 8 and 27 are the squares and cubes, respectively, of 2 and 3.

In the second of the cited paragraphs, Plato suggests that *epimorics* can be inserted between the powers of 2 and 3 ("the double and triple intervals") by use of two kinds of means, namely what we would call the *harmonic and arithmetic means*. The idea is that one mean between two given numbers (integers) p and q can be computed as follows:

if $p + r \cdot p = q - r \cdot q$ for some part r, then $p - q = r \cdot (p + q)$, and so on.

The other kind of mean between two numbers p and q can be computed as follows:

if $p + n = q - n$ for some number n, then $p - q = 2\ n$, and so on.

In Nicomachus' *Enchiridion* or 'Handbook' of harmonics (2nd century AD), Ch. 8 (Barker, GMW II, 259), the following example is used to clarify the situation:

"... A duple interval is that of 12 to 6; and it has two means, the numbers 9 and 8. Now the number 8 is a mean in harmonic proportion between 6 and 12, exceeding 6 by one third of that 6, and exceeded by 12 by one third of that 12. ... The other mean, which is 9, and which is so placed as to correspond to *paramése*, is reckoned to stand as an arithmetical mean in relation to the extremes, exceeding 6 by the same number, 3, as that by which it is exceeded by 12. ..."

The example is well chosen, because it is easy to see that

$12 = 2 \cdot 6$, $8 = 1\ 1/3 \cdot 6$, $9 = 1\ 1/2 \cdot 6$, $9 = 1\ 1/8 \cdot 8$, $12 = 1\ 1/2 \cdot 8$, and $12 = 1\ 1/3 \cdot 9$.

Every one of these relations is intimately connected with Pythagorean music theory.

The Euclidean Division of the Canon

Also the meaning of the remainder of the cited paragraphs from the *Timaeus* will become clear after the continued discussion below of important examples of Pythagorean music theory, beginning with selected propositions from the little treatise *Sectio Canonis* 'Division of the Canon', which is attributed to Euclid in several of the known sources.

***The Euclidean* Sectio Canonis (*'Division of the Canon'*)** (Barker, GMW II, 190-208)

Prop. 6. The duple interval is composed of the two greatest epimoric intervals, the hemiolic and the epitritic.

Two proofs of this proposition are given. The second, simpler proof argues as follows: If A is the hemiolic of B and B the epitritic of C, then A contains B and half of B. Therefore two A's are equal to three B's. Also B contains C and a third of C, so that three B's are equal to four C's. Therefore, two A's are equal to four C's, so that A is equal to two C's. Hence A is double G.[6]

Prop. 8. If an epitritic interval is subtracted from a hemiolic interval, the remainder is epogdoic.

In the proof, it is assumed that A is the hemiolic of B and C the epitritic of B. Then A contains B and half

[6]) Anachronistically, in terms of common fractions: if A = 3/2 B and B = 4/3 C, then A = 3/2 · 4/3 C = 2 C.

of B, so that eight A's are equal to twelve B's. Again, C contains B and a third of B so that nine G's are equal to twelve B's. Consequently, eight A's are equal to nine C's. Therefore A is equal to C and an eighth of C. Hence A is the epogdoic of C.[7]

Prop. 9. Six epogdoic intervals are greater than one duple interval.

In the proof, it is assumed that A is a number, that B is the epogdoic of A, C of B, D of G, E of D, F of E, and G of F, so that A, B, C, D, E, F, G are epogdoics of one another (a geometric progression). Then, in the least terms (see Euclid's *Elements* VIII 2),

A = 26 myriads 2,144 (the sixth power of 8),
B = A + 1/8 A = 29 myriads 4,912,
C = B + 1/8 B = 33 myriads 1,776,
D = C + 1/8 C = 37 myriads 3,248,
E = D + 1/8 D = 41 myriads 9,904,
F = E + 1/8 E = 47 myriads 2,392,
G = F + 1/8 F = 53 myriads 1,441 (the sixth power of 9).

Hence G = 53 myriads 1,441 is more than two A's = 52 myriads 4,288.[8]

Prop. 12. The octave interval is duple.

In the proof, proceeding in a not quite satisfactory way from the axiomatic assumption that concordant intervals correspond to string ratios that are either multiple or epimoric, it is observed, among other things, that the octave is made up of the hemiolic and the epitritic, the two largest epimoric intervals. It is also made up of the fifth and the fourth, and these are both epimoric (Prop. 11). Therefore, the fifth is hemiolic, the fourth is epitritic, and the octave is duple.

Prop. 13. It remains to show that the interval of a tone is epogdoic.

Indeed, if an epitritic interval is subtracted from a hemiolic interval, the remainder is epogdoic, and if a fourth is taken from a fifth, the remainder is (by definition) a tone. But the fifth is hemiolic and the fourth is epitritic. Therefore, the interval of a tone is epogdoic.

Prop. 14. The octave is less than six tones.

This follows from Prop. 9.

Prop. 15. The fourth is less than two and a half tones, and the fifth is less than three and a half tones.

This is because the octave, which is less than six tones is equal to two fourths and a tone, so that two fourths are less than five tones. And so on.

Prop. 19. To mark out the canon according to the so-called changeless system.

In the 'changeless system' of Greek music theory, a central octave is thought of as composed of two 'tetrachords' (four successive strings), separated by a tone. In Fig. 12.1 below, the two tetrachords of the central octave are called the 'middle' and the 'disjoined' tetrachord. To the central octave are joined on either side an 'extra' and an 'upper' tetrachord, and then an additional tone. The result is strings spanning a double octave.

The 'canon' in Prop. 19 is a measuring stick (a "monochord") along which a string is stretched from A to B (see again Fig. 12.1, which, by the way, is more detailed than the corresponding diagrams in the original manuscripts). A moveable bridge can take any position from E to A, defining a corresponding string length with departure from B. The positions of the bridge producing notes corresponding to various parts of the 'changeless system' are determined algorithmically, in a sequence of steps, in the following way:

1. The bass note is defined by the whole string AB. It is called *proslambanómenos*, the 'added-on,'
2. AB is divided into four equal parts, at C, D, and E. Then AB is epitritic of AC, so that *CB is a fourth above AB in pitch*. It is called the 'upper diatonic.'

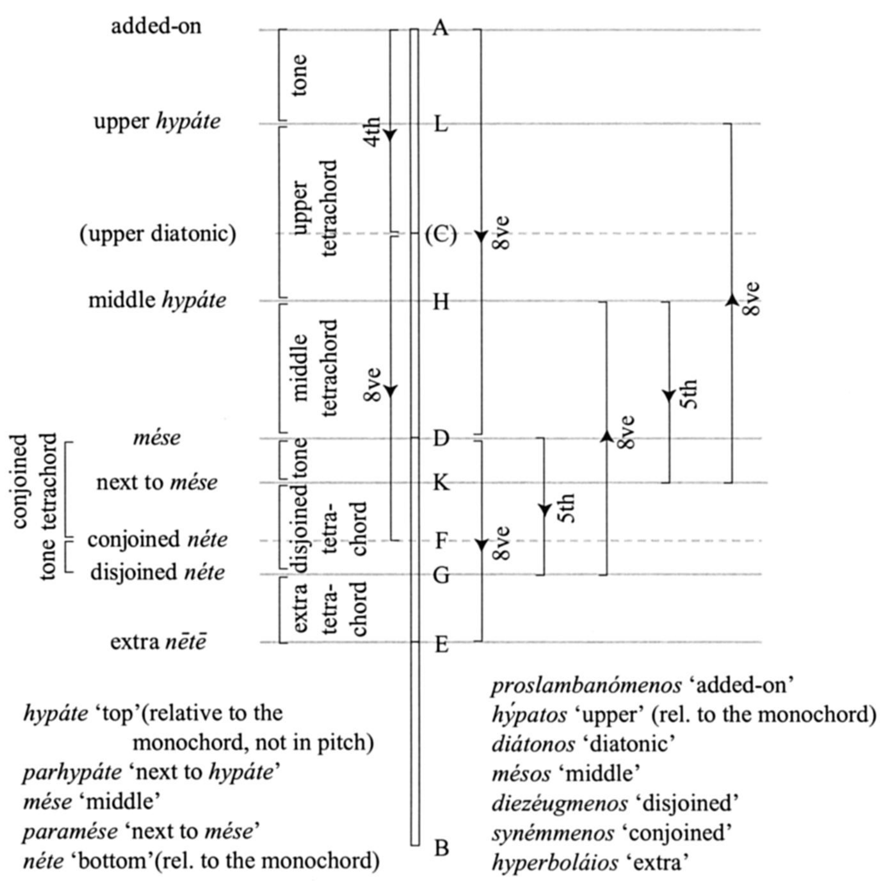


Fig. 12.1. *Sectio Canonis*, Prop. 19. Construction of the fixed notes (independent of genus).

[7]) Anachronistically: if A = 3/2 B and C = 4/3 B, then B = 3/4 C and A = 3/2 · 3/4 C = 9/8 C.

[8]) A myriad is 100 times 100. Anachronistically: G = $(9/8)^6$ · A = 531,441/262,144 A is more than 2 A.

3. AB is made duple DB. Then *DB is an octave above AB*. It is called *mése*, 'middle.'
4. AB is quadruple EB. Then *EB is two octaves above AB*. It is called 'extra *néte*' (*néte* = 'bottom').
5. CB is made duple FB. Then *FB is an octave above CB*. It is called 'conjoined *néte*.'
6. DB is made hemiolic of GB. Then *GB is a fifth above DB*. It is called 'disjoined *néte*.'
7. HB is made duple GB. Then *HB is an octave below GB*. It is called 'middle *hypáte*' (*hypáte* = 'top').
8. HB is made hemiolic of KB. Then *KB is a fifth above HB*. It is called 'next to *mése*'.
9. LB is made duple KB. Then *LB is an octave below KB*. It is called 'upper *hypáte*.'

In this way, the string lengths corresponding to all the 'fixed notes' of the 'changeless system' have been determined, and also the string length corresponding to one 'moveable note' (CB). The fixed notes bound five tetrachords and two tones of the double octave. Note that the procedure used is entirely mathematical.

The author of *Sectio Canonis* did not bother to demonstrate that the construction really yielded the desired result. It is easy to supply the missing details, for instance as follows:

10. DB is an octave below EB and a fifth below GB. Therefore, *GB is a fourth below EB*.
11. GB is an octave above HB and a fifth above DB. Therefore, *DB is a fourth above HB*.
12. HB is an octave below GB and a fifth below KB. Therefore, *KB is a fourth below GB*.
13. HB is a fifth below KB and a fourth below DB. Therefore, *DB is a tone below KB*.
14. KB is an octave above LB and a fifth above HB. Therefore, *HB is a fourth above LB*.
15. AB is an octave below DB, LB is a fourth below HB, and HB is a fourth below DB. Therefore, *AB is a tone below LB*.

Prop. 20. It remains to find the moveable notes.

The 'moveable notes' are the notes defined by seven strings *within* the four tetrachords. One of these has been found already, the 'upper diatonic.' The remaining moveable notes are determined algorithmically, as follows:

1. MB is made epogdoic of EB. Then *MB is a tone below EB*. It is called 'extra diatonic.'
2. NB is made epogdoic of MB. Then *NB is a tone below MB*. It is called 'extra *tríte*' (*tríte* = 'third').
3. XB is made epitritic of NB. Then *XB is a fourth below NB*. It is called 'disjoined *tríte*.'
4. OB is made hemiolic of XB. Then *OB is a fifth below XB*. It is called 'middle next to *hypáte*.'
5. PO is made equal to OX. Then PB is duple XB, so that *PB is an octave below XB*.
   It is called 'upper next to *hypáte*.'
6. CB is made epitritic of RB. Then *RB is a fourth above CB*. It is called 'middle diatonic.'

The distribution of tones and 'semitones' in the double octave, the "Greater Perfect System," is not explicitly mentioned by the author of *Sectio Canonis*, but it is easily determined, for instance as follows:

7. By construction, *MB is a tone below EB*, and *NB a tone below MB*.
8. GB is a fourth below EB, at the same time as NB is two tones below EB. If *the amount that GB is below NB is called a 'semitone,'* then a fourth can be divided into two tones and a semitone, where, according to *Sectio Canonis*, Prop.15, a semitone is less than half a tone.
   In terms of string ratios, GB = 1 1/3 · EB, and NB = 1 1/8 · 1 1/8 · EB. Consequently, 3 GB = 4 EB, and 64 NB = 81 EB, so that 81 · 3 GB = 81 · 4 EB = 4 · 64 NB. In other words, 243 GB = 256 NB, or GB : NB = 256 : 243. (Cf. the cited obscure passage from Plato's *Timaeus*.)
9. XB is a fourth below NB, and KB is a fourth below GB. Therefore, *KB is a semitone below XB*.
10. OB is a fifth below XB, and HB is a fifth below KB. Therefore, *HB is a semitone below OB*.
11. PB is an octave below XB, and LB is an octave below KB. Therefore, *LB is a semitone below PB*.

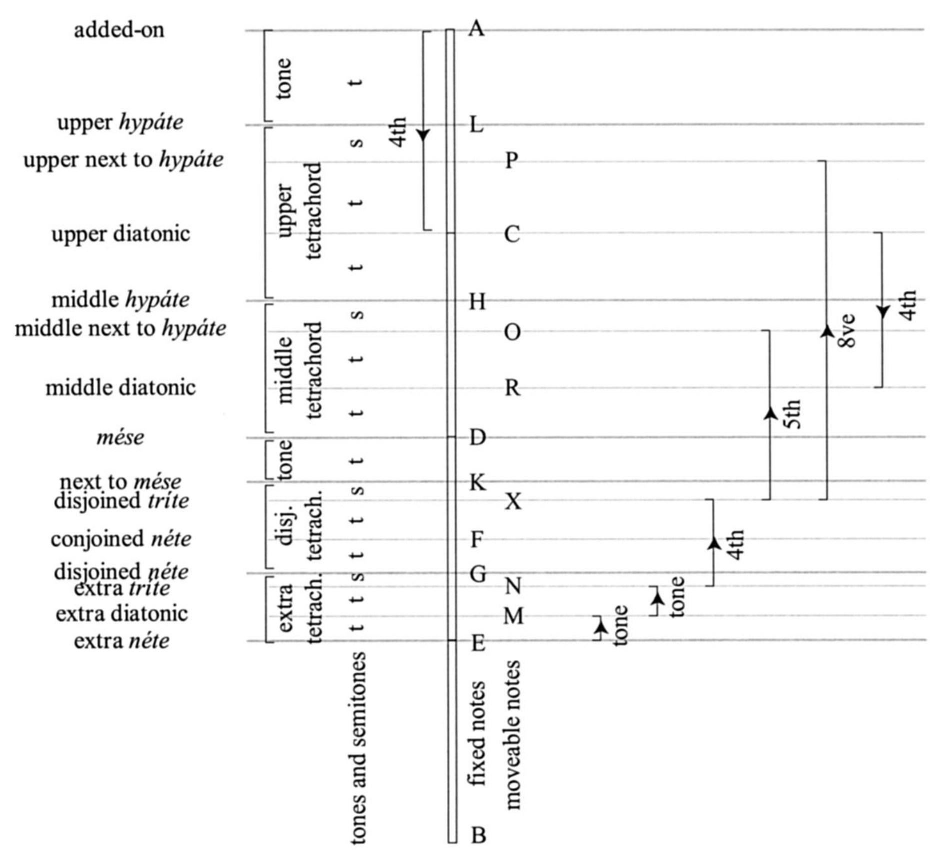


Fig. 12.2. *Sectio Canonis*, Prop. 20. Construction of the moveable notes in the diatonic genus.

12. AB is a fourth below CB, and a tone and a semitone below PB. Therefore, *PB is a tone below CB.*
13. LB is a fourth below HB, and a tone and a semitone below CB. Therefore, *CB is a tone below HB.*
14. CB is a fourth below RB, and HB a fourth below DB. Therefore, *RB is a tone below DB.* Then, *OB is also a tone below RB.*
15. CB is an octave above FB, and four tones and two semitones below XB.
    Therefore, *XB is a tone below FB.* Then, *FB is also a tone below GB.*

The distribution of tones and semitones is indicated in Fig. 12.2 above, which, by the way, is more detailed than the corresponding diagrams in the original manuscripts.[9]

Ptolemy's Construction of the Seven *tónoi* (Octave-Forms)

The distribution of tones and semitones is explicitly mentioned in the following interesting passage in Ptolemy's *Harmonics.*

Ptolemy, ***Harmonics***, **II.10-11** (Egypt, 2nd century AD; Barker, GMW II, 336ff.)

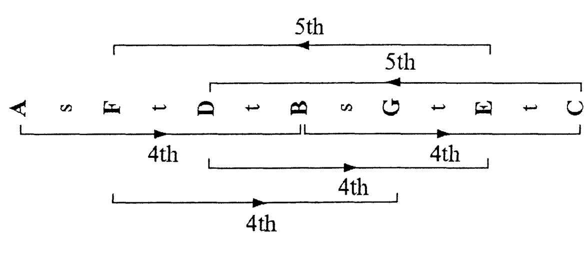
II.10 "... This (the production of modulations) can be done according to the proper method, if we begin by setting down a higher *tonos*, which we call **A**, then take first the one lower than it by a fourth, **B**, and next the one lower than **B** by a fourth, **C**, which will still be within the compass of an octave. Next, since the one lower than **C** by a fourth falls outside the octave, we take the one functionally equivalent to it, that is, the one higher than **C** by a fifth, **D**. Then, once again, we set down the one lower by a fourth than this one, **E**, and next, instead of the one lower than **E** by a fourth, since that too falls outside the octave, we make **F** the one higher than **E** by a fifth; and we set down once again the one lower than **F** by a fourth, **G**. ... It will unquestionably follow that the differences between **C** and **E**, between **G** and **E**, between **B** and **D**, and between **D** and **F** are constituted as tones, while those between **G** and **B** and **F** and **A** contain what is called the *limma*. ..."

"Now **A** corresponds to Mixolydian, **F** to Lydian, **D** to Phrygian, **B** to Dorian, **G** to Hypolydian, **E** to Hypophrygian, and **C** to Hypodorian, so that the differences between them, which have been somehow or other handed down, have now been discovered by reason."

II.11 "It is clear that in these *tonoi* that we have set out there will be, peculiar to each of them, a specific note of the octave that belongs to dynamic *mesē*, since the *tonoi* are equal in number to the species. For if we set out an octave in the intermediate range of the complete *systēma*, that is, the range from thetic middle *hypatē* to disjoined *nētē* (to allow the voice to move about and exercise itself comfortably upon melodies of middling compass, for the most part, going out infrequently to the extremes because of the hard work and force involved in slackening or tension that goes beyond the norm), the dynamic *mesē* of the Mixolydian will be attuned to the position of the disjoined next to *nētē*, so that the *tonos* may make the first species of the octave in the range set out; that of the Lydian will be attuned to the position of the disjoined *tritē*, corresponding to the second species; that of Phrygian to the position of the next to *mesē*, corresponding to the third species; that of Dorian to the position of the *mesē*, making the fourth and central species of the octave; that of Hypolydian to the position of the middle *lichanos*, corresponding to the fifth species; that of Hypophrygian to the position of the middle next to *hypatē*, corresponding to the sixth species; and that of Hypodorian to the position of the middle *hypatē*, corresponding to the seventh species. ..."

In the cited passage, Ptolemy demonstrates a simple procedure by use of which seven modulations of an initial *tónos* can be produced, each one *with its mése located within the central octave of the Greater Perfect System* (GPS), the octave most suitable for the voice. He begins with a "higher" (actually, the highest) *tónos* **A**, then considers **B** a fourth "below" **A**, and **C** a fourth below **B**. Since three fourths extend over more than one octave, a fourth *tónos* cannot be produced by going down by another fourth. Instead, the next *tónos*, called **D**, is produced by moving **C** up by a fifth. Similarly, the fifth, sixth, and seventh *tónoi*, called **E**, **F**, and **G** are produced by going down a fourth, up a fifth, and finally down again by a fourth. See Fig. 12.3 below.

Reordering the seven octave species from the "highest" to the "lowest," Ptolemy points out that **C** is now a tone below **E**, and that similarly the differences between **E** and **G**, between **B** and **D**, and between **D**



1. EC = DC – BC = 5th – 4th = tone
2. GE = FE – DE = 5th – 4th = tone
3. BG = BC – GC = 4th – ditone = *limma*
4. BC = DE = FG = AB

DB = EC = tone, FD = GE = tone,

AF = BG = *limma*

Fig. 12.3. *Harmonics* II.10. Six modulations of a higher *tónos*.

[9]) This disposition of tones and semitones (and their counterparts in other genera) is absolutely normal in Greek accounts, in Aristoxenus and his followers as well as in exponents of mathematical harmonics such as Plato, Thrasyllus, Nichomachus, Ptolemy, etc. (A. Barker, personal communication.) I want to use this opportunity to thank A. Barker for gently guiding me through some of the intricacies of Greek music theory.

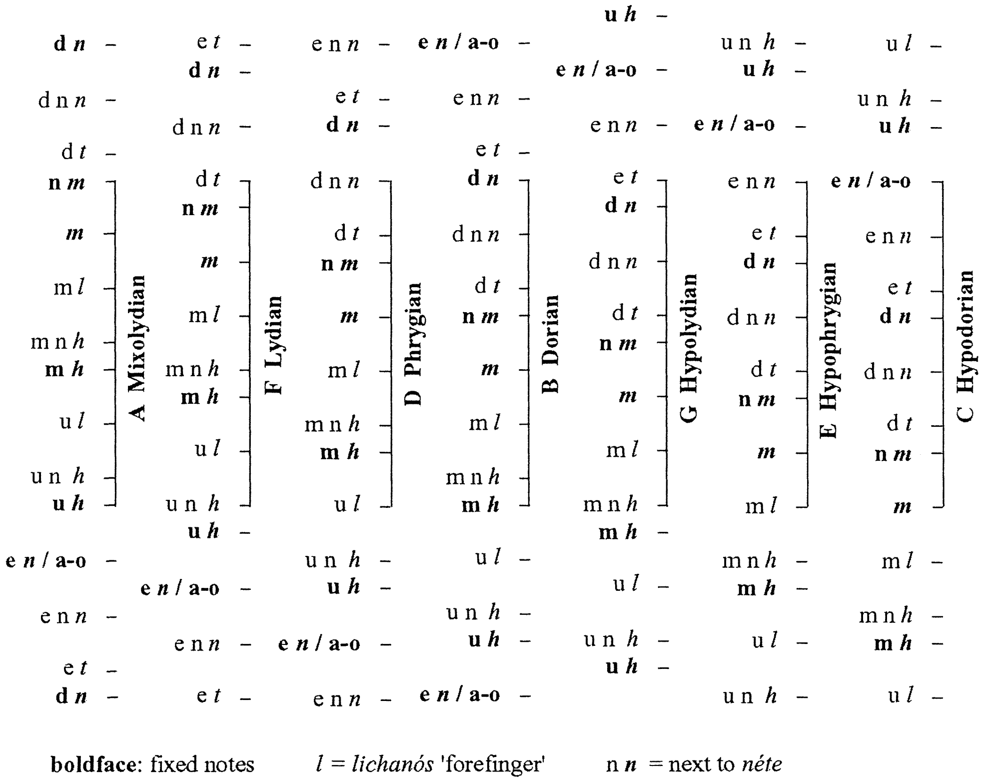


Fig. 12.4. Ptolemy's *Harmonics*, II.10-11. The seven *tónoi* explained in terms of the Greater Perfect System.

and **F**, all are a tone, while **G** is a *limma* 'remainder' (a semitone) below **B**, and **F** a *limma* below **A**.

Ptolemy then concludes that **A**, **F**, **D**, **B**, **G**, **E**, **C** are related to the seven Greek octave-forms *Mixolydian*, *Lydian*, *Phrygian*, *Dorian*, *Hypolydian* (a fourth below Lydian), *Hypophrygian* (a fourth below Phrygian), and *Hypodorian* (a fourth below Dorian).

Finally, he considers the *mése* of each one of the seven *tónoi*. (Note that if the division of the octave into intervals, called the *éidos*, meaning 'form' or 'species' of the octave, is known for a given *tónos*, then the so called "dynamic" *mése* of that *tónos* is always located between the 'higher disjunctive tone' and the tetrachord below it.) He claims that the (dynamic) *mése* of the Mixolydian corresponds to the "position" of the 'disjoined next to *néte*,' while that of the Lydian corresponds to the position of the 'disjoined *tríte*,' and so on. (What this means will be explained below.) Only in the Dorian octave-form is the dynamic *mése* located at the position of the *mése*.

In order to comprehend what is going on in *Harmonics* II,10-11, it is necessary to understand what a *tónos* is and what it means that one *tónos* is a fourth above another *tónos*. Essentially, a *tónos* is characterized by its "octave-form," the particular 'form' (*eidos*) in that *tónos* of the intervals making up an octave (with repetition of the same form in the second octave of a double octave). *The double octave is assumed to be cyclic*, in the sense that the highest and lowest notes of the double octave are considered to be identical.

Although Ptolemy does not make this clear, apparently what he means by saying that one *tónos* is a fourth, a tone, or a semitone above another one is that the octave form of the latter *tónos* is the same as the octave-form of the former *tónos*, only rotated downwards by a fourth, a tone, or a semitone.

In *Harmonics* II.11, Ptolemy explains the seven octave-forms in terms of the GPS. What he has in mind is probably something like the schematic diagram in Fig. 12.4 above (which is an elaboration of Barker's diagram in GMW II, 20, but not part of the original manuscript). In the middle column of this diagram, the fixed notes of the GPS are shown to extend downwards from **e *n*** = 'extra *néte*,' through **d *n*** = 'disjoined *néte*,' **n *m*** = 'next to *mése*,' ***m*** = '*mése*,' and **m *h*** = 'middle *hypáte*,' to **u *h*** = 'upper *hypáte*' (and **a-o** = 'added-on'). The central octave extends downwards from **d *n*** to **m *h***. It is comprised of the *disjoined tetrachord* between **d *n*** and **n *m***, the *disjunctive tone* between **n *m*** and ***m***, and the *middle tetrachord* between ***m*** and **m *h***.

In this middle column of the diagram, the names of the notes are *thetic*, meaning given '*according to position*.' In the other six columns of the diagram, the names of the notes are *dynamic*, meaning given '*according to function*.' (In the middle column, there is actually no difference between thetic and dynamic.)

The dynamic names of the notes in the first *tónos* can be thought of as being produced in the following way: The central interval remains unchanged (in position), but the GPS is moved upwards as far as possible, until **u *h*** = 'upper *hypáte*' takes the place of the thetic **m *h*** = 'middle *hypáte*.' (The 'added-on' note is disregarded.) Since the GPS is moved as high as possible in the first *tónos*, this is also called the "highest" *tónos*. In the other six *tónoi*, the GPS is moved downwards, one step at a time, until it reaches its lowest possible position, with **e *n*** = 'extra *néte*' taking the place of thetic **d *n*** = 'disjoined *néte*.' In this process, as pointed out by Ptolemy, the dynamic **mése** moves from the position of the moveable note d n *n* = 'disjoined next to *néte*' to the position of the fixed note **m *h*** = 'middle *hypáte*.'

Note that Ptolemy's construction in *Harmonics* II.10-11 is *independent of genus*. What counts is only the positions of the fixed notes making up the boundaries of the tetrachords and of the disjunctive tones. Thus, for instance, the octave-form of Ptolemy's "higher" *tónos* **A** (Mixolydian) is composed, in descending order, of a tone and two tetrachords. This is what Ptolemy elsewhere (*Harmonics* II.3; Barker, GMW II, 322-323) calls the "first" octave-form. The other six octave-forms are obtained from this first one by rotating the first octave-form downwards, one step at a time.

Ptolemy's Tables of Numbers for the Seven tónoi *in Several Familiar Genera*

Harmonics II.15 (Barker, GMW II, 352-355) contains a series of numerical tables giving an "exposition of the numbers that make up the divisions of the familiar genera in the seven *tonoi*." It is not very difficult to see how the tables were constructed. First, the numbers (relative string lengths) for the boundaries of the fixed octave are chosen to be 60 for the 'disjoined *néte*' and 2 · 60 = 120 for the 'middle *hypáte*.' Then the corresponding number for the dynamic *mése* in the Mixolydian, a tone below the upper boundary of the octave, is 60 · 1 1/8 = 67 1/2. The *mése* in the Lydian is a semitone below that, *etc*. Thus, in Ptolemy's tables the numbers associated with the dynamic *mése* in each one of the seven *tónoi* are as follows, independent of genus. (Note the use of *sexagesimal fractions*, *rounded to the first sixtieth* in the tables.)

60	· 1 1/8 = 67 1/2	=		67;30	ξζ λ′	(Mixolydian)
67 1/2	· 256/243 = 71 1/9	=	(appr.)	71;07	οα′ ζ′	(Lydian)
71 1/9	· 1 1/8	=		80	π′	(Phrygian)
80	· 1 1/8	=		90	φ′	(Dorian)
90	· 256/243 = 94 22/27	=	(appr.)	94;49	φδ′ μθ	(Hypolydian)
94 22/27	· 1 1/8 = 106 2/3	=		106;40	ϱς′ μ′	(Hypophrygian)
106 2/3	· 1 1/8	=		120	ϱϰ′	(Hypodorian)

The five genera considered in Ptolemy's tables are 1) (a mixture of) "tense chromatic" and "tonic diatonic," 2) "soft diatonic" and "tonic diatonic," 3) "tonic diatonic," 4) "tonic diatonic" and "ditonic diatonic," 5) "tonic diatonic" and "tense diatonic." In tonic diatonic, for instance, the tetrachords are divided into intervals with the epimoric ratios 1 1/8, 1 1/7, 1 1/27. Here, of course, 1 1/8 · 1 1/7 · 1 1/27 = (9/8 · 8/7 · 28/27 = 4/3 =) 1 1/3. Therefore, in particular, in Ptolemy's table 11.2 (Lydian), column 3 (tonic diatonic), the numbers are constructed as follows, with departure from Lydian *mése* = 71 1/9:

d *t*	121;54 · 1/2	= 60;57	ξ′ νζ′
n *m*	60;57 · 1 1/27	= 63;13	ξγ′ ιγ′
m	71 1/9	= 71;07	οα′ ζ′
m *l*	71 1/9 · 1 1/8	= 80	π′
m n *h*	80 · 1 1/7	= 91;26	φα′ ϰς′
m *h*	91;26 · 1 1/27	= 94;49	φδ′ μθ′
u *l*	94;49 · 1 1/8	= 106;40	ϱς′ μ′
u n *h*	106;40 · 1 1/7	= 121:54	ϱϰα′ νδ′.

In the exhibited example the computed numbers are not, as would have been expected, strictly contained between 60 and 120, the chosen numbers for the boundaries of the fixed central octave. Actually, an inspection of all the seven tables in *Harmonics* II.15 reveals that the computed numbers stay between the expected boundaries for all considered genera only in the cases of Mixolydian (**A**), Dorian (**B**), and Hypodorian (**C**). These are precisely the cases when the boundaries of the fixed octave coincide with fixed notes of the *tónoi*.

Hypothetical Tables of String Ratios for the Seven Babylonian Diatonic Modes

Nothing corresponding to the Greek identification of concordant string pairs with epimoric ratios of string lengths is known (*so far*) *from any cuneiform texts*. On the other hand, this fact is quite surprising, in view of the enthusiastic calculations with all kinds of numbers and measures that are so characteristic for many kinds of both Sumerian and Babylonian cuneiform texts. It is, therefore, an interesting thought experiment to try to figure out what Babylonian mathematicians/musicians could have made of the idea of epimoric string ratios if they had known about it.

In the first of the star diagrams in Fig. 10.7, the one for the *išartu* mode, the first side of the star defines a descending fifth from string 2 to string 6, the next side an ascending fourth from string 6 to string 3, the third side a descending fifth from string 3 to string 7, and so on. Suppose now that the (relative) length of string 2 is 1. Since string 6 is reached from string 2 by a *descending fifth*, its relative length is (theoretically) 1 · 1 1/2 = 1 1/2, or simply 3/2. Since string 3 is reached from string 6 by an *ascending fourth*, its relative length is (3/2)/(1 1/3) = 3/2 · 3/4 = 9/8, or simply, in modern notation, $3^2/2^3$. And so on. In other words, the ratios of string lengths in the *išartu* mode can be computed as follows:

string	ratio
2	1 = 1
6	1 · 3/2 = 3/2
3	3/2 · 3/4 = $3^2/2^3$
7	$3^2/2^3$ · 3/2 = $3^3/2^4$
4	$3^3/2^4$ · 3/4 = $3^4/2^6$
8	$3^4/2^6$ · 3/2 = $3^5/2^7$
5	$3^5/2^7$ · 3/4 = $3^6/2^9$

In the second of the star diagrams in Fig. 10.7, the one for the *qablītu* mode, the tightened string 5 T is reached from string 2 by a *descending fourth.* Therefore the corresponding string ratio is 1 · 4/3 = 4/3 = $2^2/3$. Similarly in the third star diagram, the one for

	B 2	A# 3 T	A 3	G# 4 T	G 4	F# 5 T	F 5	E# 6 T	E 6	D# 7 T	D 7	C# 8 T	C 8	B 9
išartu	1		$3^2/2^3$		$3^4/2^6$		$3^6/2^9$		3/2		$3^3/2^4$		$3^5/2^7$	2
qablītu	"		"		"	$2^2/3$			"		"		"	"
niš tuḫri	"		"		"	"			"		"	$2^4/3^2$		"
nīd qabli	"		"	$2^5/3^3$		"			"		"	"		"
pītu	"		"	"		"			"	$2^7/3^4$		"		"
embūbu	"	$2^8/3^5$		"		"			"	"		"		"
kitmu	"	"		"		"		$2^{10}/3^6$		"		"		"
išartu	"	"		"		"		"				"		"

Fig. 12.5. Hypothetical string ratios for the seven Babylonian (ditonic) diatonic modes.

the *niš tuḫri* mode, the tightened string 8 T is reached from 5 T by another *descending fourth.* Therefore, the corresponding string ratio for string 8 T is $2^2/3 \cdot 4/3 = 2^4/3^2$. In the same way, the string ratio is $2^4/3^2 \cdot 2/3 = 2^5/3^3$ for string 4 T, it is $2^5/3^3 \cdot 4/3 = 2^7/3^4$ for string 7 T, it is $2^7/3^4 \cdot 2/3 = 2^8/3^5$ for string 3 T, and it is $2^8/3^4 \cdot 4/3 = 2^{10}/3^6$ for string 6 T.

The result of this series of computations is displayed in tabular form below.

There is no doubt that Ptolemy must have been familiar with the numbers in this table, which is like the tables in *Harmonics* II.15, but in the case of the ditonic diatonic genus. Incidentally, the numbers which Ptolemy associated in his tables with the dynamic *mése* in each one of the seven octave-forms are the numbers associated above with strings 3, 4 T, 5 T, 6, 7 T, 8 T, and 9. Cf. Fig. 12.8 below.

As is well known, an important role was played in Babylonian mathematics by so called "regular sexagesimal numbers" defined as numbers containing no other factors than positive or negative powers of 2, 3, or 5. Regular sexagesimal numbers have the important property that both they and their reciprocal numbers can be expressed as integers divided by suitable powers of the base 60. Interestingly, but purely by coincidence, all the hypothetical string ratios for the seven Babylonian diatonic heptatonic modes displayed above in Fig. 12.5 are regular sexagesimal numbers. In Fig. 12.6, all those string ratios are written first as ratios of powers of 2 and 3, then as common fractions, and finally as sexagesimal numbers, both in the Babylonian form and in the form used by Ptolemy in his tables.

Ptolemy's Construction of the Seven tónoi *in the Case of the Ditonic Diatonic Genus*

Ptolemy does not explain what is the origin of the Greater Perfect System, why the seven *tónoi* can be obtained by means of the construction in Fig. 12.3 above, and why the numbers for the octave-forms of the mentioned "familiar genera" do not stay strictly between the expected boundaries 60 and 120. The common answer to these questions is that all the various genera appearing in ancient Greek music theory are simply modifications of one *basic genus*, the so called "ditonic" diatonic, in which all intervals are made up of tones and semitones, in particular the *tetrachords of two tones* (*a ditone*) *and a semitone.* This is the genus of the scale constructed in *Sectio Canonis*, Prop. 20 (Fig. 12.2 above), which in its turn is closely related to the seven Old Babylonian diatonic modes.

In the case of the ditonic diatonic genus, an octave-form is a particular distribution of tones and semitones within an octave or double octave. It is instructive to see how the seven octave-forms can be generated in a surprisingly simple way in this special case.

Now, return to Ptolemy's result in *Harmonics* II.10 that the seven *tónoi* he had constructed (and their corresponding octave-forms), namely **A**, **F**, **D**, **B**, **G**, **E**, **C**, in this order, exceed each other by a semitone, two tones, a semitone, and two tones. *Take it for*

(ditonic) diatonic string ratios

	common fractions		sexagesimal fractions	as in *Harmonics* II.15
2 (B)	1	1	1	60
3 T (A#)	$2^8/3^5$	256/243	1;03 12 35 33 20	63;13
3 (A)	$3^2/2^3$	9/8	1;07 30	67;30
4 T (G#)	$2^5/3^3$	32/27	1;11 06 40	71;07
4 (G)	$3^4/2^6$	81/64	1;15 56 15	75;56
5 T (F#)	$2^2/3$	4/3	1;20	80
5 (F)	$3^6/2^9$	729/512	1;25 25 46 52 30	85;26
6 T (E#)	$2^{10}/3^6$	1024/729	1;24 16 47 24 26 40	84;17
6 (E)	3/2	3/2	1;30	90
7 T (D#)	$2^7/3^4$	128/81	1;34 48 53 20	94;49
7 (D)	$3^3/2^4$	27/16	1;41 1 5	101;15
8 T (C#)	$2^4/3^2$	16/9	1;46 40	106;40
8 (C)	$3^5/2^7$	243/128	1;53 54 22 30	113;54
9 (B)	2	2	2	120

Fig. 12.6. (Ditonic) diatonic string ratios ordered by size.

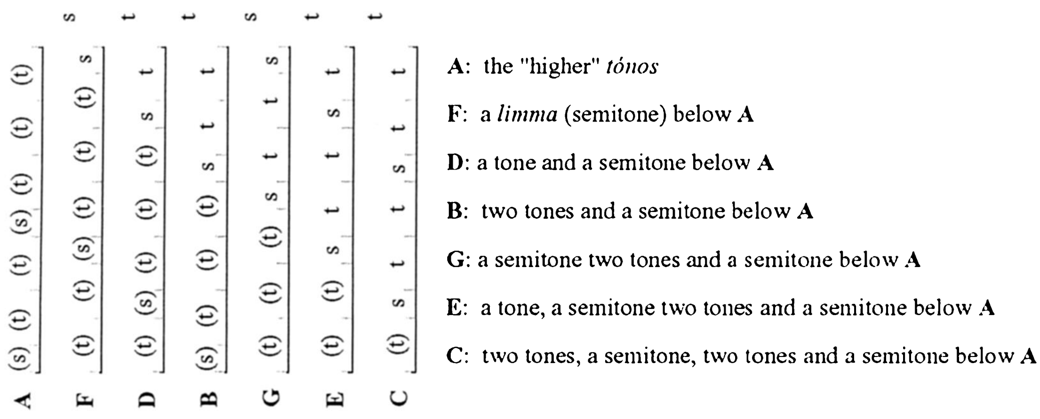


Fig. 12.7. Generation of the seven octave-forms in the ditonic diatonic genus. (Not in *Harmonics* II.10.)

then string 3, a fifth above string 7, and finally string 6, a fourth below string 3. *In this whole retuning algorithm, strings 2 and 9 are not changed.*

In Ptolemy's procedure in *Harmonics* II.10, on the other hand, limited to the case of the ditonic diatonic genus, the seven Greek octave-forms are constructed by starting with the Mixolydian octave-form and then rotating that octave-form twice by a fourth downwards, then by a fifth upwards, by a fourth downwards, by a fifth upwards, and by a fourth downwards. *In this whole generating algorithm, the boundaries of the octave are not changed.*

granted that the octave-forms stay strictly within the boundaries of the prescribed interval. Then the octave-form **F**, being a semitone below **A**, must begin with a semitone. See Fig. 12.7 above. Similarly, the octave-form **D**, being a tone below **F**, hence a semitone and a tone below **A**, must begin with a tone and a semitone. And so on. The final octave-form, **C**, must begin with two tones, a semitone, two tones, and a semitone, together four tones and two semitones. Clearly, to complete the octave, the last interval of the octave-form **C** must then be a tone. Now, when the whole octave-form **C** is known, the whole octave-form **E**, being a tone above **C** will also be known. And so on. See again Fig. 12.7. Thus, the layout of the seven octave-forms in the ditonic diatonic genus is completely determined by the restriction to a fixed octave, together with Ptolemy's assumption in *Harmonics* II.10 that they are dependent on each other as in Fig. 12.3.

The distribution of tones and semitones (in descending order) in the seven Greek ditonic diatonic octave-forms (as in Fig. 12.7 above) can be compared with the corresponding distribution of tones and semitones (*from string 2 to string 9*) in the seven Old Babylonian diatonic modes (as in Fig. 10.7 above). The proposed identifications are as follows:

Mixolydian	t t t s t t s	*išartu* ('normal')
Lydian	s t t t s t t	*embūbu*
Phrygian	t s t t t s t	*nīd qabli*
Dorian	t t s t t t s	*qablītu* ('middle')
Hypolydian	s t t s t t t	*kitmu*
Hypophrygian	t s t t s t t	*pītu*
Hypodorian	t t s t t s t	*nīš tuḫri*

According to the retuning algorithm in UET VII 74+, § 2, the seven Old Babylonian diatonic modes can be constructed by starting with a *sammû* instrument tuned to the *išartu* mode, then successively tightening first string 5, a fourth below string 2, then strings 8, a fourth below string 5 (and the equivalent string 1, a fifth above string 5), then string 4, a fifth above string 8, then string 7, a fourth below string 4,

The obvious similarity between the two algorithmic procedures immediately suggests that the Old Babylonian retuning algorithm and Ptolemy's generating algorithm must be mathematically equivalent in some sense. A simple way of demonstrating this equivalence is by use of 7/3 star diagrams as in Fig. 12.8 below. Indeed, start with the first 7/3 star diagram, which is in the configuration of the *išartu* mode, corresponding to **A** Mixolydian. In this configuration, the octave extending from string 2, corresponding to thetic **n *m*** = disjoined *néte*, to string 9, corresponding to thetic **u *h*** = middle *hypáte*, is divided, in descending order, into a tone and two tetrachords. In particular, there is a tritone between strings 2 and 5. The second star diagram in Fig. 12.8 is in the configuration of the *qablītu* mode, corresponding to **B** Dorian. It is the result of a rotation by a fourth downwards of the configuration in the first start diagram. That rotation moves, in particular, the semitone between strings 1 (8) and 2 (9) to a semitone between strings 4 and 5, and the semitone between strings 5 and 6 to a semitone between 1 (8) and 2 (9). In other words, the combined effect of the rotation is that it changes the semitone between strings 5 and 6 to a semitone between strings 4 and 5. Therefore, the only observable result of the rotation is that it tightens string 5, which is the same as moving **n *m*** = next to *mése* a semitone upwards. In the same way, the only observable result of moving the configuration in the second star diagram by a fourth downwards is that string 1 (8) is tightened. Another rotation four steps downwards leads to the configuration of the *nīš tuḫri* mode (**C** = Hypodorian). Next, a rotation five steps upwards (to the left) leads to the configuration of the *nīd qabli* mode (**D** Phrygian),

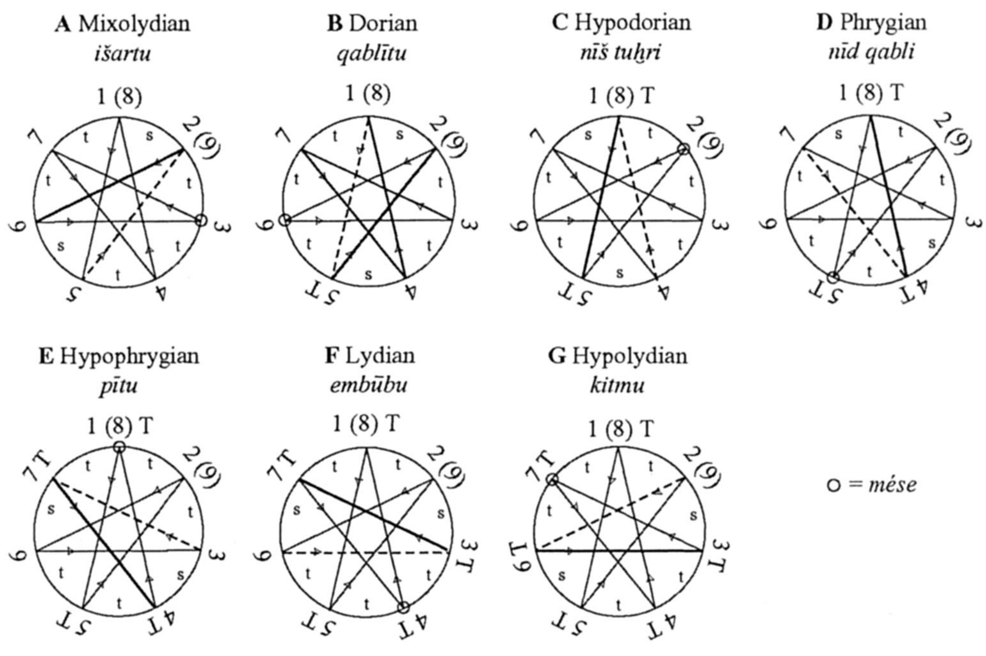


Fig. 12.8. Ptolemy's construction of the seven Greek octave-forms, explained in terms of 7/3 star diagrams.

a rotation four steps downwards leads to the *pītu* mode (**E** Hypophrygian), a new rotation five steps upwards leads to the *embūbu* mode (**F** Lydian). After the sixth rotation, the final configuration is in the *kitmu* mode, corresponding to **G** Hypolydian, with all strings except string 2 (9) tightened. *This means that in this whole retuning algorithm, interpreted as successive tightenings of strings, strings 2* (thetic disjoined *néte*) *and 9* (thetic middle *hypáte*) *are never affected.* That is as it should be, because all the seven octave-forms are supposed to stay strictly within the *fixed central octave* of the GPS.

(Alternatively, following UET VII 74+, § 3, the seven Old Babylonian diatonic modes can be constructed as in Fig. 10.5 by starting again with the *sammû* instrument tuned to the *išartu* mode, then successively *loosening* strings 2, 6, 3, 7, 4, 1, and 8. *In this alternative retuning algorithm, only string 5, halfway between strings 1 and 9, is never affected.*)

13. *Cyclic Representations of Modes in a Medieval Islamic Manuscript*[10]

Perhaps the most influential of all medieval Islamic treatises on music was *Kitāb al-Adwār* (*The Book of Cycles*) by Ṣafī ad-Dīn al-Urmawi († 1294). A study of a preliminary version of that work was published by Wright in BSOAS 58. (See also Manik, ATM and Wright, *Modal Systems*.) A partial French translation of the work is contained within the translation of a commentary on it in D'Erlanger, MA 3. A beautifully written copy of *Kitāb al-Adwār*, manuscript ljs235 in the L. J. Schoenberg Collection, is available online at http://dewey.library.upenn.edu/sceti/ljs. This copy of the work contains diagrams presented with an exceptionally high degree of care, precision and visual clarity.

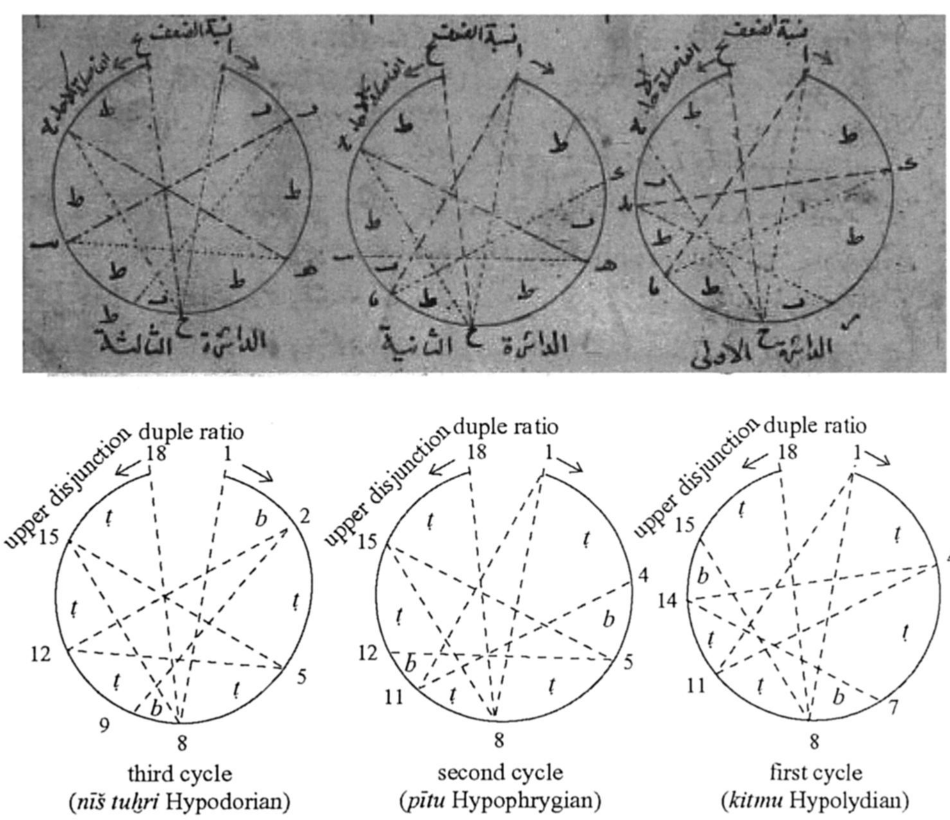


Fig. 13.1. *Kitāb al-Adwār*, Ch. 6. Cyclic representations of three diatonic modes. The detail from the manuscript is published here with the kind permission of L. J. Schoenberg.

[10]) I want to thank Anne Kilmer and Janet Smith, who told me about the existence of star-figures illustrating a 16th century Arabic musical tuning system.

Of particular interest in connection with the discussion above of the 7/3 star diagram on CBS 1766 and its conjectured use in Babylonian music theory are six circular diagrams on p. 8 of the manuscript ljs235. Three of those diagrams are reproduced in Fig. 13.1 above, together with line drawings showing the same three circles but with English translations of the Arabic text in and around the original diagrams.

Along the outside of the periphery of each one of the three circles eight numbers in alphabetic *abjad* numerals denote eight notes within an octave. Along the insides of the peripheries, the Arabic letters *ṭ* and *b* denote tones and semitones, respectively.

The so called *abjad* numerals are based on an older form of the Arabic alphabet, which begins with the letters *a*, *b*, *j*, *d*. The first 9 letters stand for the ones, from 1 to 9, the next 9 letters stand for the tens, from 10 to 90, and so on. In *Kitāb al-Adwār*, 17 fixed "Pythagorean" notes within an octave are denoted by the numbers from 1 to 17 in *abjad* notation. The positions of the notes within the octave are as shown below. (Cf. Manik, ATM, 54-56).

1	2	(3)	4	5	(6)	7	8	9	(10)	11	12	(13)	14	15	16	(17)	18
C	Db	(Ebb)	D	Eb	(Fb)	E	F	Gb	(Abb)	G	Ab	(Bbb)	A	Bb	B	(Dbb)	C′
	s	s	c	s	s	c	s	s	s	c	s	s	c	s	s	s	c

Here s stands for a limma or semitone (string ratio 256/243), while c stands for a "Pythagorean comma" (string ratio 531441/524288). Note that a whole tone can be divided into two semitones and a comma. It is easy to check that the notes and numbers within brackets above do not appear in the cyclic representations of the three diatonic modes in Fig. 13.1.

Ṣafī ad-Dīn's 17 notes were constructed by use of an algorithm resembling the algorithm used in the *Sectio Canonis*, Props. 19-20 (Figs. 12.1-2 above), in terms of only *octaves*, *fifths*, *fourths*, and *whole tones*. See Manik, ATM, Ch. 3, and Fig. 13.2 below.

With Ṣafī ad-Dīn's notations, the eight notes and seven intervals of the cyclic diagrams in Fig. 13.1 are from left to right along the peripheries of the circles, in *descending* order:

first circle:	18	(t)	15	(s)	14	(t)	11	(t)	8	(s)	7	(t)	4	(t)	1
	C′		Bb		A		G		F		E		C		C
second circle:	18	(t)	15	(t)	12	(s)	11	(t)	8	(t)	5	(s)	4	(t)	1
	C′		1Bb		Ab		G		F		Eb		D		C
third circle:	18	(t)	15	(t)	12	(t)	9	(s)	8	(t)	5	(t)	2	(s)	1
	C′		Bb		Ab		Gb		F		Eb		Db		C

This means that in each one of the three cases the octave is divided, in descending direction, into a whole tone, the "upper disjunction," followed by two consecutive identical ditonic diatonic tetrachords. Note that in contrast to the diagrams in Fig. 10.7, where all the sections of the peripheries of the circles are of equal length, the sections of the peripheries of the circles in Fig. 13.1 corresponding to the tones are larger than the sections corresponding to the semitones. Moreover, the first and last of the notes indicated in the cyclic diagrams in Fig. 13.1 are an octave apart, but in contrast to the identical representations of strings 1 and 8 in the 7/3 star figure on CBS 1766, in the diagrams in Fig. 13.1 the first and eighth notes are separated by a gap called 'relationship of the double,' meaning "duple ratio" (ratio of the octave).

Another difference between the diagrams in Fig. 10.7 and those in Fig. 13.1 is that in the latter ones the unclear dichords are not represented by sides of the star diagrams. Thus, in the first diagram in Fig. 13.1, the unclear dichord could have been indicated by a dashed straight line connecting the third note to the seventh note, and so on.[11]

In spite of the mentioned differences between the diagrams in Fig. 13.1 and those in Fig. 10.7, it is clear that they are basically of the same type. However, it would be difficult to believe that there is any historical connection between the Old Babylonian star diagram in CBS 1766 and the more elaborate cyclic diagrams in *Kitāb al-Adwār*.

The three remaining cyclic diagrams on p. 8 of the Schoenberg copy of *Kitāb al-Adwār*, called 'fourth cycle,' 'fifth cycle,' and 'sixth cycle' are of the same general type, but represent three modes of a different genus.

For comparison, here are the eight notes and seven intervals along the peripheries of each one of these three additional cyclic diagrams:

fourth circle:	18	(t)	15	(c, s)	13	(s, s)	11	(t)	8	(c, s)	6	(s, s)	4	(t)	1
	C′		Bb		Bbb		G		F		Fb		D		C
fifth circle:	18	(t)	15	(t)	12	(c, s)	10	(s, s)	8	(t)	5	(c, s)	3	(s, s)	1
	C′		Bb		Ab		Abb		F		Eb		Ebb		C
sixth circle:	18	(t)	15	(c, s)	13	(t)	10	(s, s)	8	(c, s)	6	(t)	3	(s, s)	1
	C′		Bb		Bbb		Abb		F		Fb		Ebb		C

[11]) In BSOAS 58, 467, Wright explains the situation by writing that "the notes of the scale are marked around the circumference and lines are drawn across to link those which are a fourth or fifth apart in order to show the number of consonant intervals each scale contains." This is also the interpretation given in a medieval Arabic commentary to *Kitāb al-Adwār* written by Mubārak Šāh. (See D'Erlanger, MA III, 324-333.) My sincere thanks are due to O. Wright for helping me to understand some difficulties in this text.

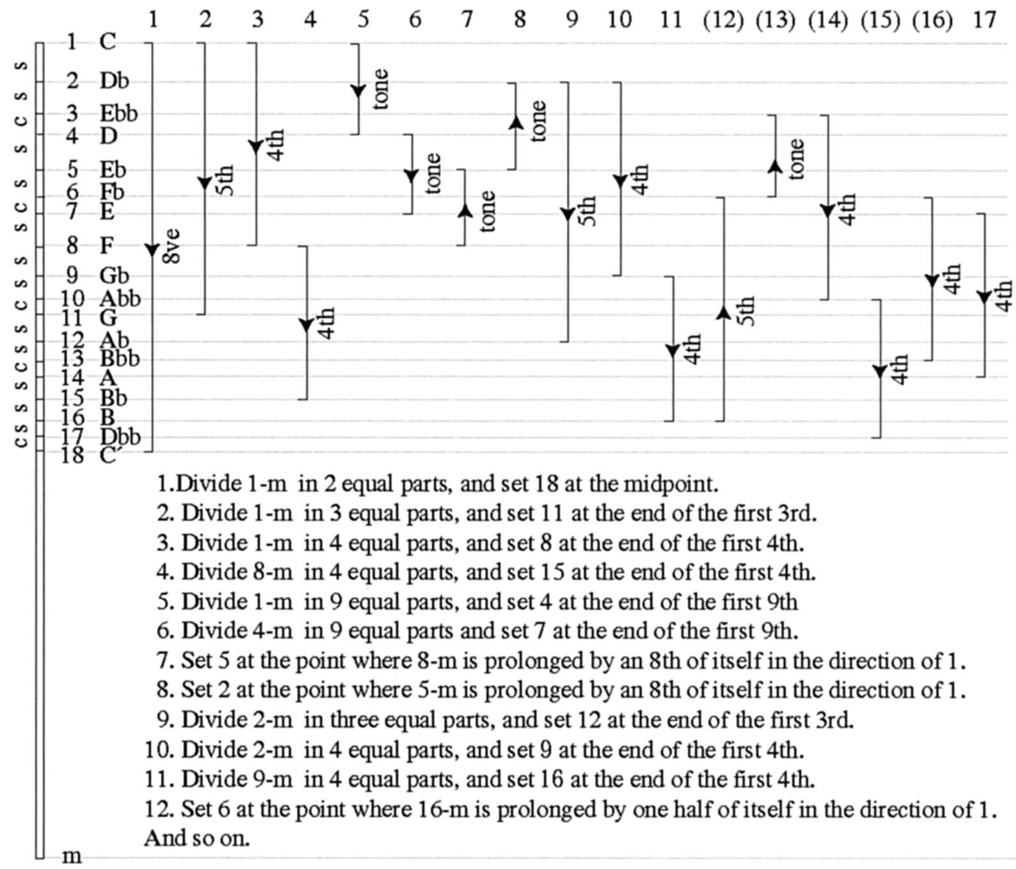


Fig. 13.2. The algorithm for Ṣafī ad-Dīn's division of the octave into 17 intervals.

Here c, s stands for an "apotome," a comma followed by a limma, with the string ratio 256/243 · 65536/59049 = 2187/2048, while s, s stands for a "double limma," with the string ratio 256/243 · 256/243 = 65536/59049. Both kinds of intervals are indifferently written as j in the diagrams, while the limma and the comma both are written as b.

Abbreviations and Bibliography

Abbreviations, if not listed here, follow the housestyle of *AfO*.

appr. = approximately
csm = common seed measure
GPS = Greater Perfect System

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